Local Fractional Fokker-Planck Equation

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We propose a new class of differential equations, which we call local fractional differential equations. They involve local fractional derivatives and appear to be suitable to deal with phenomena taking place in fractal space and time. A local fractional analog of the Fokker-Planck equation has been derived starting from the Chapman-Kolmogorov condition. We solve the equation with a specific choice of the transition probability and show how subdiffusive behavior can arise. [S0031-9007(97)04966-1]

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In recent studies of scaling phenomena [1–3,4], many applications for derivatives and integrals of fractional order have been found. The main aim of most of these papers is to formulate fractional integrodifferential equations to describe some scaling process. Modifications of equations governing physical processes such as the diffusion equation, the wave equation, and the Fokker-Planck equation have been suggested [5–10] which incorporate fractional derivatives with respect to time. Recently, Zaslavsky [11] argued that the chaotic Hamiltonian dynamics of particles can be described by using fractional generalization of the Fokker-Planck-Kolmogorov (FPK) equation. However, fractional derivatives are nonlocal and hence such equations are not suitable for the study of local scaling behavior. In the present work we rigorously derive fractional analogs of equations like the FPK equation involving one space variable. Our approach differs from the above mentioned ones since we use local fractional Taylor expansion, which was established only recently [12]. As is argued below, such equations can provide appropriate schemes for describing evolutions (e.g., sub- or superdiffusive) normally not obtained from the usual FPK equation.

It was realized recently [12] that there is a direct quantitative connection between fractional differentiability properties of continuous but nowhere differentiable functions and the dimensions of their graphs. In order to show this, a new notion of local fractional derivative (LFD) was introduced. The LFD of order q of a function $f(y)$ was defined by

$$\mathbb{D}^q f(y) = \lim_{x\to y} \frac{d^q[f(x) - f(y)]}{[d(x-y)]^q}, \qquad 0 < q \le 1, \quad (1)$$

where the derivative on the right-hand side (RHS) is the Riemann-Liouville fractional derivative [13,14], viz., for $0 < q < 1$

$$\frac{d^q f(x)}{[d(x-a)]^q} = \frac{1}{\Gamma(1-q)} \frac{d}{dx} \int_a^x \frac{f(y)}{(x-y)^q}\, dy . \quad (2)$$

As is obvious from Eq. (2), the operator $d^q/[d(x-a)]^q$ is nonlocal, and, further, the $d^q f(x)/[d(x-a)]^q \neq 0$ for $f(x) = \text{const}$. The motivation for the definition of $\mathbb{D}^q$ was to correct for both of these features. It was shown in [12], in particular, that the LFD of Weierstrass nowhere differentiable function exists up to (critical) order $1-\gamma$, where $1+\gamma$ $(0<\gamma<1)$ is the box dimension of the graph of the function. Further, the use of LFD to study pointwise behavior of multifractal functions was also demonstrated. The definition was generalized [15] for a function for which the first N derivatives exist by replacing $[f(x) - f(y)]$ on the RHS of Eq. (1) by

$$\tilde{F}_N(x,y) = f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(y)}{\Gamma(n+1)} (x-y)^n, \quad (3)$$

with q in the interval $(N, N+1]$. Sometimes it is essential to distinguish between limits taken from above and below. In that case we define

$$\mathbb{D}^q_{\pm} f(y) = \lim_{x\to y^{\pm}} \frac{d^q \tilde{F}_N(x,y)}{[d\,\pm(x-y)]^q} . \quad (4)$$

We will assume $\mathbb{D}^q = \mathbb{D}^q_+$ unless mentioned otherwise. We note that when $q = n$, an integer $\mathbb{D}^q f(y)$ is simply $d^n f(y)/dy^n$.

The importance of the above definition also lies in the fact that the LFDs appear naturally in the fractional Taylor expansion as the coefficient of the power with fractional exponent. Thus, for $\Delta = x - y$

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(y)}{\Gamma(n+1)} \Delta^n + \frac{\mathbb{D}^q_{\pm} f(y)}{\Gamma(q+1)} (\pm\Delta)^q + R_q(y,\Delta), \quad (5)$$

where $R_q(y,\Delta)$ is the remainder [12].

The basic idea of the present paper is to utilize such fractional Taylor expansions in the Chapman-Kolmogorov condition and obtain analogs of the FPK equation. We begin by recalling the usual procedure and difficulties of obtaining the FPK equation. Let $W(x,t)$ denote the probability density for a random variable X taking value x at time t, then

$$W(x, t+\tau) = \int P(x, t+\tau | x', t) W(x', t)\, dx', \quad (6)$$

where $P(x_1, t_1 | x_2, t_2)$ denotes the transition probability from x_1 at time t_1 to x_2 at time t_2 and $\tau \ge 0$. The usual

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FPK equation is obtained [16] from Eq. (6) by expanding the integrand in a Taylor series.

There are a number of limitations of this approach arising naturally from the assumptions going into its derivation. For instance, as noted in [17], probability distributions whose second moment does not exist are not described by the FPK equation even though such distributions may satisfy the original Chapman-Kolmogorov equation. Also, as emphasized in [18], the differentiability assumption may also break down in various situations. For instance, the transitional probability density may not be differentiable at $x = x'$, in which case the derivation of the FP equation itself will break down. Another situation is when we have a fractal function as the initial probability density. In such a case even the usual Fokker-Planck operator cannot be operated on the initial density.

It is thus of interest to broaden the class of differential equations one can derive starting from the Chapman-Kolmogorov equation and to study various processes described by them. In this paper we pursue the possibility of removing the assumption of differentiability of probability densities. We follow the usual procedure to derive the Fokker-Planck equation from Eq. (6), except that we now expand the integrand using fractional Taylor expansion (5) instead of ordinary Taylor expansion. Thus, if $\Delta = x - x'$,

$$\begin{aligned} W(x,t+\tau) = W(x,t) &+ \sum_{n=1}^{N} \frac{1}{\Gamma(n+1)} \left(\frac{\partial}{\partial(-x)}\right)^n \int dx'\, \Delta^n P(x+\Delta, t+\tau | x,t) W(x,t) \\ &+ \frac{1}{\Gamma(\beta+1)} \mathbb{D}^{\beta}_{x-} \left[\int_x^{\infty} dy\, (y-x)^{\beta} P(y,t+\tau|x,t) W(x,t) \right] \\ &+ \frac{1}{\Gamma(\beta+1)} \mathbb{D}^{\beta}_{x+} \left[\int_{-\infty}^{x} dy\, (x-y)^{\beta} P(y,t+\tau|x,t) W(x,t) \right] + \text{remainder}, \end{aligned} \tag{7}$$

where $\mathbb{D}_x$ is a partial LFD with respect to x. Now if $0 < \alpha \le 1$

$$W(x,t+\tau) - W(x,t) = \frac{\tau^{\alpha} \mathbb{D}^{\alpha}_t W(x,t)}{\Gamma(\alpha+1)} + \text{remainder},$$

where $\mathbb{D}_t$ is a partial LFD with respect to t. In general, α and β may depend on x and t. But we assume that α and β are constants. Therefore we get

$$\begin{aligned} \frac{\tau^{\alpha} \mathbb{D}^{\alpha}_t W(x,t)}{\Gamma(\alpha+1)} &= \sum_{n=1}^{N} \left(\frac{\partial}{\partial(-x)}\right)^n \left[\frac{M_n(x,t,\tau)}{\Gamma(n+1)} W(x,t) \right] \\ &+ \mathbb{D}^{\beta}_{x-} \left[\frac{M^{+}_{\beta}(x,t,\tau)}{\Gamma(\beta+1)} W(x,t) \right] \\ &+ \mathbb{D}^{\beta}_{x+} \left[\frac{M^{-}_{\beta}(x,t,\tau)}{\Gamma(\beta+1)} W(x,t) \right], \end{aligned} \tag{8}$$

where

$$M^{+}_{a}(x,t,\tau) = \int_x^{\infty} dy\, (y-x)^a \times P(y,t+\tau|x,t) \qquad a > 0, \tag{9}$$

$$M^{-}_{a}(x,t,\tau) = \int_{-\infty}^{x} dy\, (x-y)^a \times P(y,t+\tau|x,t) \qquad a > 0, \tag{10}$$

and

$$M_a(x,t,\tau) = M^{+}_{a}(x,t,\tau) + M^{-}_{a}(x,t,\tau) \tag{11}$$

are transitional moments. The limit $\tau \to 0$ gives us an equation

$$\mathbb{D}^{\alpha}_t W(x,t) \equiv \mathcal{L}(x,t) W(x,t), \tag{12}$$

where the operator $\mathcal{L}$ is given by

$$\begin{aligned} \mathcal{L}(x,t) = \sum_{n=1}^{N} \left(\frac{\partial}{\partial(-x)}\right)^n A^n_{\alpha}(x,t) &+ \mathbb{D}^{\beta}_{x-} A^{\beta}_{\alpha-}(x,t) \\ &+ \mathbb{D}^{\beta}_{x+} A^{\beta}_{\alpha+}(x,t), \end{aligned} \tag{13}$$

where

$$A^{\beta}_{\alpha\mp}(x,t) = \lim_{\tau\to 0} \frac{M^{\pm}_{\beta}(x,t,\tau)\Gamma(\alpha+1)}{\tau^{\alpha}\Gamma(\beta+1)} \tag{14}$$

and

$$A^{\beta}_{\alpha}(x,t) = A^{\beta}_{\alpha+}(x,t) + A^{\beta}_{\alpha-}(x,t). \tag{15}$$

Here corresponding A_{α}'s are assumed to exist. We would like to point out that Eq. (12) is analogous to truncated Kramers-Moyal expansion. Two rather important special cases are $0 < \beta < 1$ and $1 < \beta < 2$. In the former case we get the operator

$$\mathcal{L}(x,t) = \mathbb{D}^{\beta}_{x-} A^{\beta}_{\alpha-}(x,t) + \mathbb{D}^{\beta}_{x+} A^{\beta}_{\alpha+}(x,t),$$

and in the latter case we get

$$\begin{aligned} \mathcal{L}(x,t) = -\frac{\partial}{\partial x} A^1_{\alpha}(x,t) &+ \mathbb{D}^{\beta}_{x-} A^{\beta}_{\alpha-}(x,t) \\ &+ \mathbb{D}^{\beta}_{x+} A^{\beta}_{\alpha+}(x,t). \end{aligned}$$

This operator can be identified as generalizations of the Fokker-Planck operator in one space variable. It is clear that when $\alpha = 1$ and $\beta = 2$ we get back the usual Fokker-Planck operator.

It may be pointed out that the local fractional differential equations (LFDE) that we are proposing here are a new kind of differential equations. To our knowledge this is the first direct occurrence of such equations. We note that they are different from the conventional fractional differential equations which have been studied to some extent in the literature [13,14] and which have found several applications ranging from solutions of Bessel equation, diffusion on curved surfaces to wave equation etc. In fact, the equations appearing in [1,2,5–9,11] are all conventional fractional differential equations. On the other hand, the present LFDE involve operators $\mathbb{D}^q$, which found successful applications [12] in studying differentiability properties of nowhere differentiable functions and relating them to dimensions. They are appropriate to address scaling phenomena. It is for this reason that one would expect the equations governing the fractal processes to be LFDE. At this stage it is worth reflecting for a moment on the behavior of meaningful solutions of simple LFDE. We begin by considering the equation

$$\mathbb{D}_x^q f(x) = g(x). \tag{16}$$

The questions of the general conditions guaranteeing solutions of such an equation is an involved one. We note that the equation $\mathbb{D}_x^q f(x) = \text{const}$ does not have a finite solution when $0 < q < 1$. Interestingly, the solutions to (16) can exist when $g(x)$ has a fractal support. For instance, when $g(x) = \chi_C(x)$, the membership function of a cantor set C [i.e., $g(x) = 1$ if x is in C and $g(x) = 0$ otherwise], the solution with initial condition $f(0) = 0$ exists if $q = \alpha \equiv \dim_H C$. Explicitly, generalizing the Riemann integration procedure,

$$f(x) \equiv \frac{P_C(x)}{\Gamma(\alpha + 1)} = \lim_{N\to\infty} \sum_{i=0}^{N-1} \frac{(x_{i+1} - x_i)^\alpha}{\Gamma(\alpha + 1)} F_C^i, \tag{17}$$

where x_i are subdivision points of the interval $[x_0 = 0, x_N = x]$ and F_C^i is a flag function which takes value 1 if the interval $[x_i, x_{i+1}]$ contains a point of the set C and 0 otherwise. Note that $P_C(x)$ is a Lebesgue-Cantor (staircase) function and satisfies the bounds $ax^\alpha \le P_C(x) \le bx^\alpha$ where a and b are suitable positive constants. In general, the algorithm of Eq. (17) will work only for the sets C for which $\dim_B C = \dim_H C$ (in fact in this case only N^α terms in the summation are nonzero). More details about solutions of such equations and algorithms will be discussed elsewhere [19].

Returning back to Eq. (14) it is clear that the small time behavior of different transitional moments decide the order of the derivative with respect to time (in order to demonstrate this point we consider the example of a Lévy process below). On the other hand, small distance behavior of transitional probability or the differentiability property of the initial probability density would dictate the order of space derivative. Depending on the actual values of α and β as well as their interrelation the above local FFPK equation will describe different processes.

Equations which give rise to an evolution-semigroup [i.e., corresponding to evolution operators P_t satisfying $P_0 = \text{identity}$ and $P_{t+s} = P_t \circ P_S, s, t \ge 0$, as in Eq. (27) below] are of interest in physics. Equation (12) corresponds to a semigroup if $\alpha = 1$. One can then write down a formal solution of the above equation in this case as follows. In the time independent case we have

$$W(x,t) = e^{\mathcal{L}(x)t} W(x,0), \tag{18}$$

and when the operator depends on time we have

$$W(x,t) = \overleftarrow{T} e^{\int_0^t \mathcal{L}(x,t')\,dt'} W(x,0), \tag{19}$$

where $\mathcal{L}$ is an operator in Eq. (12) and $\overleftarrow{T}$ is the time ordering operator.

For the symmetric stable Lévy process of index μ, the moments scale as $M_\gamma(\lambda t) = \lambda^{\gamma/\mu} M_\gamma(t)$ and we get

$$\mathbb{D}_t^{\gamma/\mu} W(x,t) = \mathbb{D}_{x-}^\gamma [A_{\gamma/\mu-}^\gamma(x,t) W(x,t)] + \mathbb{D}_{x+}^\gamma [A_{\gamma/\mu+}^\gamma(x,t) W(x,t)]. \tag{20}$$

Since the process is symmetric the first derivative does not appear. The order of the time derivative depends on that of the space derivative but it is always less than 1. Now there is only one free parameter γ which is restricted to the range $0 < \gamma < \mu$. In this case the value of γ will be decided by the differentiability class of the initial distribution function. (The details and intricacies will be addressed in [19].) When $\mu = 2$ and $\gamma = 2$ we get back the usual Fokker-Planck equation describing a Gaussian process. Equation (20) forms one example where the usual derivation of the FPK equation breaks down and we get nontrivial values for the orders of the derivatives.

As our next example, we consider the transition probability

$$P(x, t + \tau | x', t) = \frac{1}{\sqrt{\pi \Delta P(t,\tau)}} e^{-(x-x')^2/\Delta P_C(t,\tau)} \tag{21}$$

$$= \delta(x - x') \quad \text{if } \Delta P_C(t,\tau) = 0, \tag{22}$$

where $\Delta P_C(t,\tau) = P_C(t + \tau) - P_C(t)$. This transition probability describes a nonstationary process which corresponds to transitions occurring only at times which lie on a fractal set. Such a transition probability can be used to model phenomena where transition is very rare, for instance, diffusion in the presence of traps. The second moment is given, from Eq. (11), by

$$M_2(t,\tau) = \frac{\Delta P_C(t,\tau)}{2} \simeq \frac{1}{2} \frac{\mathbb{D}^\alpha P_C(t)}{\Gamma(\alpha + 1)} \tau^\alpha = \frac{\tau^\alpha}{2} \chi_C(t). \tag{23}$$

This gives us the following local fractional Fokker-Planck equation (in this case an analog of a diffusion equation):

$$\mathbb{D}_t^\alpha W(x,t) = \frac{\Gamma(\alpha + 1)}{4} \chi_C(t) \frac{\partial^2}{\partial x^2} W(x,t). \tag{24}$$

We note that even though the variable t is taking all real positive values the actual evolution takes place only for values of t in the fractal set C. The solution of Eq. (24) can be easily obtained as

$$W(x,t) = P_{t-t_0} W(x,t_0), \tag{25}$$

where

$$P_{t-t_0} = \lim_{N\to\infty} \prod_{i=0}^{N-1} \left[1 + \frac{1}{4} (t_{i+1} - t_i)^\alpha F_C^i \frac{\partial^2}{\partial x^2} \right]. \tag{26}$$

The above product converges because except for the number of terms of order N^α all other terms take value 1. It is clear that for $t_0 < t' < t$

$$W(x,t) = P_{t-t'} P_{t'-t_0} W(x,t_0) \tag{27}$$

and P_t gives rise to a semigroup evolution. Using Eq. (17) it can be easily seen that

$$W(x,t) = e^{[P_C(t)/4](\partial^2/\partial x^2)} W(x,t_0 = 0)\,. \tag{28}$$

Now choosing $W(x,0) = \delta(x)$ and using the Fourier representation of delta function then we get the solution

$$W(x,t) = \frac{1}{\sqrt{\pi P_C(t)}}\, e^{-x^2/P_C(t)}. \tag{29}$$

Its consistency can easily be checked by substituting this in Chapman-Kolmogorov equation. This solution satisfies the bounds

$$\frac{1}{\sqrt{\pi b t^\alpha}}\, e^{-x^2/a t^\alpha} \le W(x,t) \le \frac{1}{\sqrt{\pi a t^\alpha}}\, e^{-x^2/b t^\alpha} \tag{30}$$

for some $0 < a < b$. This is a model solution of a subdiffusive behavior. It is clear that when $\alpha = 1$ we get back the typical solution of the ordinary diffusion equation which is $(\pi t)^{-1/2} \exp(-x^2/t)$.

To conclude, we have derived the generalization of the FP equation which involves the local fractional derivatives. Our equations are fundamentally different from any of the equations proposed previously since they involve LFDs. They are local and more natural generalizations of ordinary differential equations. LFDEs deserve a separate study in their own right [19]. We would like to point out that LFDEs naturally give rise to dynamical systems of a new kind (neither discrete nor continuous) in which time evolution takes place for values of time belonging to a Cantor-like set. We further remark that since the present analog of the FPK equation is derived from first principles, we feel that our equation will have general applicability in the field of physics. We expect them to be of value in the studies of anomalous diffusion, chaotic Hamiltonian systems, disordered phenomenon, etc. In our derivation we assumed that the orders α and β of derivatives involved are constants. This would require a modification for the description of multiscaling multifractal processes. We further note that directional LFDs are defined in [20]. Using them it may be possible to obtain the local FPK equation involving several variables.

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