Nonlinear Modes of Liquid Drops as Solitary Waves

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The nonlinear dynamic equations of the surface of a liquid drop are shown to be directly connected to Korteweg–de Vries (KdV) systems, giving traveling solutions that are cnoidal waves. They generate multiscale patterns ranging from small harmonic oscillations (linearized model), to nonlinear oscillations, up through solitary waves. These non-axis-symmetric localized shapes are also described by a KdV Hamiltonian system. Recently such "rotons" were observed experimentally when the shape oscillations of a droplet became nonlinear. The results apply to droplike systems from cluster formation to stellar models, including hyperdeformed nuclei and fission. [S0031-9007(98)05553-7]

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A fundamental understanding of nonlinear oscillations of a liquid drop (NLD), which reveals new phenomena and flows more complicated than linear theory suggests, is needed in diverse areas of science and technology. Besides their direct use in rheological and surfactant theory [1–7], such models apply to cluster physics [8], super- and hyperdeformed nuclei [1], nuclear breakup and fission [2,3,8], thin films [9], radar [4], and even stellar masses and supernova [1,10]. Theoretical approaches are usually based on numerical calculations within different NLD models [2–4], and explain/predict axis-symmetric, nonlinear oscillations that are in very good agreement with experiment [1,5–7]. However, there are experimental results which show non-axis-symmetric modes, for example, traveling rotational shapes [5,6] that can lead to fission, cluster emission, or fusion [5–7].

In this Letter the existence of analytic solutions of NLD models that give rise to traveling solutions which are solitary waves is proven. Second order nonlinear terms in the deviation of the shape from a sphere produce surface oscillations that are cnoidal waves [11]. By increasing the amplitude of these oscillations, the nonlinear contribution grows and the drop's surface, under special conditions (nonzero angular momentum), can transform from a cnodial wave form into a solitary wave. This same evolution can occur if there is a nonlinear coupling between the normal modes. Thus this approach leads to a unifying dynamical picture of such modes; specifically, the cnoidal solution simulates harmonic oscillations developing into anharmonic ones, and under special circumstances these cnoidal wave forms develop into solitary waves. Of course, in the linear limit the theory reproduces the normal modes of oscillation of a surface.

Two approaches are used: Euler equations [2,3], and Hamiltonian equations, which describe the total energy of the system [2]. We investigate finite amplitude waves, for which the relative amplitude is smaller than the angular half-width. These excitations are also "long" waves, important in the cases of externally driven systems, where the excited wavelength depends on the driving frequency. The first original observations of traveling waves on liquid drops are described in [5]. Similar traveling or running waves are also discussed or quoted in [2,6]. These results suggest that higher amplitude nonlinear oscillations can lead to a traveling wave that originates on the drop's surface and develops towards the interior. This is shown to be related in a simple way to special solitary wave solutions, called "rotons" in the present analysis. Recent experiments and numerical tests [8,12] suggest the existence of stable traveling waves for a nonlinear dynamics in a circular geometry, reenforcing the theory.

A new NLD model for describing an ideal, incompressible fluid drop exercising irrotational flow with surface tension, is employed in the analysis. Series expansion in terms of spherical harmonics is replaced by localized, nonlinear shapes shown to be analytic solutions of the system. The flow is potential and therefore governed by Laplace's equation for potential flow, $\Delta\Phi = 0$, while the dynamics is described by Euler's equation,

$$\rho[\partial_t \vec{v} + (\vec{v} \cdot \nabla)\vec{v}] = -\nabla P + \vec{f}\,, \tag{1}$$

where P is pressure. If the density of the external force field is also potential, $\vec{f} = -\nabla\Psi$, where Ψ is proportional to the potential (gravitational, electrostatic, etc.), then Eq. (1) reduces to Bernoulli's scalar equation. The boundary conditions (BC) on the external free surface of the drop, $\Sigma 1$, and on the inner surface $\Sigma 2$ [2,3,11], are $\dot{r}|_{\Sigma 1} = (r_t + r_\theta \dot{\theta} + r_\phi \dot{\phi})|_{\Sigma 1}$ and $\dot{r}|_{\Sigma 2} = 0$, respectively. $\Phi_r = \dot{r}$ is the radial velocity, $\Phi_\theta = r^2\dot{\theta}$, $\Phi_\phi = r^2 \sin\theta\,\dot{\phi}$ are the tangential velocities. The second BC occurs only in the case of fluid shells or bubbles. A convenient geometry places the origin at the center of mass of the distribution $r(\theta, \phi, t) = R_0[1 + g(\theta)\eta(\phi - Vt)]$ and introduces for the dimensionless shape function $g\eta$ a variable denoted ξ. Here R_0 is the radius of the undeformed spherical drop and V is the tangential velocity of the traveling solution ξ moving in the ϕ direction and having a constant transversal profile g in the θ direction. The linearized form of the first BC, $\dot{r}|_{\Sigma 1} = r_t|_{\Sigma 1}$, allows only radial vibrations and no tangential motion of the fluid on $\Sigma 1$ [2,3,11]. The second BC restricts the radial flow to a spherical layer of depth $h(\theta)$ by requiring

$\Phi_r|_{r=R_0-h} = 0$. This condition stratifies the flow in the surface layer, $R_0 - h \leq r \leq R_0(1 + \xi)$, and the liquid bulk $r \leq R_0 - h$. In what follows the flow in the bulk will be considered negligible compared to the flow in the surface layer. This condition does not restrict the generality of the argument because h can always be taken to be R_0. Nonetheless, keeping $h < R_0$ opens possibilities for the investigation of more complex fluids, e.g., superfluids, flow over a rigid core, multilayer systems [2,7] or multiphases, etc. Instead of an expansion of Φ in terms of spherical harmonics, consider the following form:

$$\Phi = \sum_{n=0}^{\infty}(r/R_0 - 1)^n f_n(\theta, \phi, t). \quad (2)$$

The convergence of the series is controlled by the value of the small quantity $\epsilon = \max|\frac{r-R_0}{R_0}|$ [11]. The condition $\max|h/R_0| \simeq \epsilon$ is also assumed to hold in the following development. Laplace's equation introduces a system of recursion relations for the functions f_n, $f_2 = -f_1 - \Delta f_0/2$, etc., where Δ_Ω is the (θ, ϕ) part of the Laplacean. Hence the set of unknown f_n's reduces to f_0 and f_1. The second BC, plus the condition $\xi_\phi = -V\xi_t$, for traveling waves, yields to second order in ϵ,

$$f_{0,\phi} = VR_0^3 \sin^2\theta\, \xi(1 + 2\xi)/h + \mathcal{O}_3(\xi), \quad (3)$$

i.e., a connection between the flow potential and the shape, which is typical of nonlinear systems. Equation (3) together with the relations $f_1 \simeq R_0^2\xi_t \simeq \frac{2h}{R_0} f_2 \simeq -\frac{h\Delta f_0}{R_0+2h}$, which follow from the BC and recursion, characterize the flow as a function of the surface geometry. The balance of the dynamic and capillary pressure across the surface $\Sigma 1$ follows by expanding up to third order in ξ the square root of the surface energy of the drop [2,3,11],

$$U_S = \sigma R_0^2 \int_{\Sigma 1}(1 + \xi) \times \sqrt{(1 + \xi)^2 + \xi_\theta^2 + \xi_\phi^2/\sin^2\theta}\, d\Sigma, \quad (4)$$

and by equating its first variation with the local mean curvature of $\Sigma 1$ under the restriction of volume conservation. The surface pressure, in third order, reads

$$P|_{\Sigma 1} = \frac{\sigma}{R_0}(-2\xi - 4\xi^2 - \Delta_\Omega\xi + 3\xi\xi_\theta^2 ctg\theta), \quad (5)$$

where σ is the surface pressure coefficient and the terms $\xi_{\phi,\theta}$, $\xi_{\phi,\phi}$, and $\xi_{\theta,\theta}$ are neglected because the relative amplitude of the deformation ϵ is smaller than the angular half-width L, $\xi = \xi_{\phi\phi} \simeq \epsilon^2/L^2 \ll 1$, as most of the experiments [6,7,9,12] concerning traveling surface patterns show. Equation (5) plus the BC yield, to second order in ϵ,

$$\Phi_t|_{\Sigma 1} + \frac{V^2R_0^4\sin^2\theta}{2h^2}\xi^2 = \frac{\sigma}{\rho R_0}(2\xi + 4\xi^2 + \Delta_\Omega\xi - 3\xi^2\xi_\theta ctg\theta). \quad (6)$$

The linearized version of Eq. (6) together with the linearized BC, $\Phi_r|_{\Sigma 1} = R_0\xi_t$, yield a limiting case of the model, namely, the normal modes of oscillation of a liquid drop with spherical harmonic solutions [2,3]. Differentiation of Eq. (6) with respect to ϕ together with Eqs. (3) and (5) yields the dynamical equation for the evolution of the shape function $\eta(\phi - Vt)$:

$$A\eta_t + B\eta_\phi + Cg\eta\eta_\phi + D\eta_{\phi\phi\phi} = 0, \quad (7)$$

which is the Korteweg–de Vries (KdV) equation [8,11] with coefficients depending parametrically on θ

$$\begin{aligned} A &= \frac{R_0^2(R_0 + 2h)\sin^2\theta}{h}, \\ B &= -\frac{\sigma}{\rho R_0}\frac{(2g + \Delta_\Omega g)}{g}, \\ C &= 8\left(\frac{V^2R_0^4\sin^4\theta}{8h^2} - \frac{\sigma}{\rho R_0}\right), \\ D &= -\frac{\sigma}{\rho R_0\sin^2\theta}. \end{aligned} \quad (8)$$

In the case of a two-dimensional liquid drop, the coefficients in Eq. (8) are all constant. Equation (7) has traveling wave solutions in the ϕ direction if $Cg/(B - AV)$ and $D/(B - AV)$ do not depend on θ. These two conditions introduce two differential equations for $g(\theta)$ and $h(\theta)$ which can be solved with the boundary conditions $g = h = 0$ for $\theta = 0, \pi$. For example, $h_1 = R_0\sin^2\theta$ and $g_1 = P_2^2(\theta)$ is a particular solution which is valid for $h \ll R_0$. It represents a soliton with a quadrupole transverse profile, which is in good agreement with [2,6]. The next higher order term in Eq. (6), $-3\xi^2\xi_\theta ctg\theta$, introduces a $\eta^2\eta_\phi$ nonlinear term into the dynamics and transforms the KdV equation into the modified KdV equation [11]. The traveling wave solutions of Eq. (7) are then described by the Jacobi elliptic function (sn) [11]

$$\eta = \alpha_3 + (\alpha_2 - \alpha_3)\,\mathrm{sn}^2 \times \left(\sqrt{\frac{C(\alpha_3 - \alpha_2)}{12D}}(\phi - Vt); m\right), \quad (9)$$

where the α_i are the constants of integration introduced through Eq. (7) and are related through the velocity $V = C(\alpha_1 + \alpha_2 + \alpha_3)/3A + B/A$ and $m^2 = \frac{\alpha_3-\alpha_2}{\alpha_3-\alpha_1}$. $m \in [0, 1]$ is the free parameter of the elliptic sn function. This result for Eq. (9) is known as a cnoidal wave solution with angular period $T = K[m]\sqrt{C(\alpha_3 - \alpha_1)/3D}$, where $K(m)$ is the Jacobi elliptic integral. If $\alpha_2 \to \alpha_1 \to 0$, then $m \to 1, T \to \infty$ and a one-parameter (η_0) family of traveling pulses (solitons or antisolitons) is obtained,

$$\eta_{sol} = \eta_0\,\mathrm{sech}^2[(\phi - Vt)/L], \quad (10)$$

with velocity $V = \eta_0 C/3A + B/A$ and angular half-width $L = \sqrt{12D/C\eta_0}$. Taking for the coefficients A to D the values given in Eq. (8) for $\theta = \pi/2$ (the equatorial cross section) and h_1, g_1 from above, one can calculate numerical values of the parameters of any roton excitation function of η_0 only.

The soliton, among other wave patterns, has a special shape-kinematic dependence, $\eta_0 \simeq V \simeq 1/L$; a higher soliton is narrower and travels faster. This relation can be used to experimentally distinguish solitons from other modes or turbulence. When a layer thins ($h \rightarrow 0$) the coefficient C in Eq. (8) approaches zero on average, producing a break in the traveling wave solution (L becomes singular) because of the change of sign under the square root, Eq. (9). Such wave turbulence from capillary waves on thin shells was first observed in [9]. For the water shells described there, Eq. (8) gives $h(\mu m) \leq 20\nu/k$, that is $h = 15$–$25\ \mu$m at $V = 2.1$–$2.5\ \mathrm{ms}^{-1}$ for the onset of wave turbulence, in good agreement with the abrupt transition experimentally noticed. The cnoidal solutions provide the nonlinear wave interaction and the transition from competing linear wave modes ($C \leq 0$) to turbulence ($C \simeq 0$). In the KdV Eq. (7), the nonlinear interaction balances or even dominates the linear damping and the cnoidal (roton) mode occurs as a bend mode (h small and coherent traveling profile) in agreement with [9]. The condition for the existence of a positive amplitude soliton is $gCD \geq 0$ which, for $g \leq 0$, limits the velocity from below to the value $V \geq h\omega_2/R_0$, where ω_2 is the Lamb frequency for the $\lambda = 2$ linear mode [2,3]. This inequality can be related to the "independent running wave" described in [6], which lies close to the $\lambda = 2$ mode. Moreover, since the angular group velocity of the (λ, μ) normal mode, $V_{\lambda,\mu} = \omega_\lambda/\mu$, has practically the same value for $\lambda = 2$ ($\mu = 0, \pm 1$, tesseral harmonics) and for $\lambda = \mu$, any λ (sectorial harmonics) this inequality seems to be essential for any combination of rank 2 tesseral or sectorial harmonics, in good agreement with the conclusions in [2]. The periodic limit of the cnoidal wave is reached for $m \simeq 0$, that is, $\alpha_2 - \alpha_3 \simeq 0$, and the shape is characterized by harmonic oscillations [sn $\rightarrow$ sin in Eq. (9)] which realize the quadrupole mode of a linear theory, Y_2^μ limit [2,3], or the oscillations of tesseral harmonics [2] (Fig. 1).

The NLD model introduced in this paper yields a smooth transition from linear oscillations to solitary traveling solutions (rotons) as a function of the parameters α_i, namely, a transition from periodic to nonperiodic shape oscillations. In between these limits the surface is described by nonlinear cnoidal waves. In Fig. 1 the transition from a periodic limit to a solitary wave is shown, in comparison with the corresponding normal modes which can initiate such cnoidal nonlinear behavior. This situation is similar to the transformation of the flow field from periodic modes at small amplitude to traveling waves at larger amplitude [2,6]. The solution goes into a final form if the volume conservation restriction is enforced: $\int_\Sigma [1 + g(\theta)\eta(\phi, t)]^3\, d\Omega = 4\pi$ and requires $\eta(\phi, t)$ to be periodic. The periodicity condition, $nK[(\alpha_3 - \alpha_2)/(\alpha_3 - \alpha_1)] = \pi\sqrt{\alpha_3 - \alpha_1}$ for any positive integer n, is fulfilled only for a finite number of n values, and hence a finite number of corresponding cnoidal modes. In the roton limit the periodicity condition becomes a quasiperiodic one because the amplitude decays rapidly. This approach could be extended to describe elastic modes of surface as well as their nonlinear coupling to capillary waves. The double-periodic structure of the elliptic solutions [11] could describe the new family of normal wave modes predicted in [4].

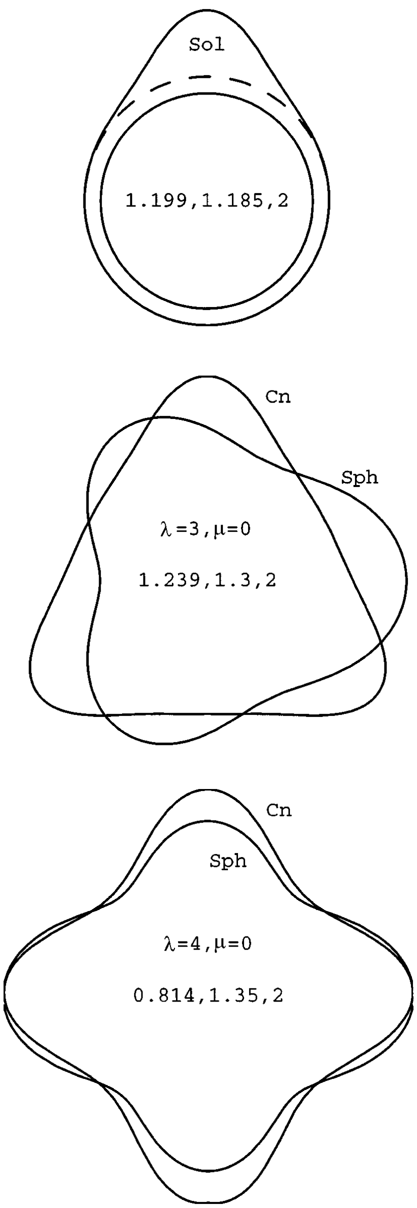

FIG. 1. The cnoidal solution for $\theta = 0$. The soliton limit and a three- and four-mode solution is shown. The closest spherical harmonics to each of the cnoidal wave profiles (labeled Cn and Sol, respectively) is given for comparison. The labels λ, μ, and the parameters $\alpha_{1,2,3}$ of the coresponding cnoidal solution are given.

The development up to this point was based on Euler's equation. The same result will now be shown to emerge from a Hamiltonian analysis of the NLD system. Recently, Natarajan and Brown [2] showed that the NLD is a Lagrangian system with the volume conservation condition being a Lagrange multiplier. In the third order deviation from spherical, the NLD becomes a KdV infinite-dimensional Hamiltonian system described by a nonlinear Hamiltonian function $H = \int_0^{2\pi} \mathcal{H}\, d\phi$. In the linear approximation, the NLD is a linear wave Hamiltonian system [2,3]. If terms depending on θ are absorbed into definite integrals (becoming parameters) the total energy is a function of η only. Taking the kinetic energy from [2,3], Φ from Eq. (2), and using the BC, the dependence of the kinetic energy on the tangential velocity along the θ direction, Φ_θ, becomes negligible and the kinetic energy can be expressed as a $T[\eta]$ functional. For traveling wave solutions $\partial_t = -V\partial_\phi$, to third order in ϵ, after a tedious but feasible calculus, the total energy is

$$E = \int_0^{2\pi} (C_1\eta + C_2\eta^2 + C_3\eta^3 + C_4\eta_\phi^2)\, d\phi\,, \quad (11)$$

where $C_1 = 2\sigma R_0^2 S_{1,0}^{1,0}$, $C_2 = \sigma R_0^2(S_{1,0}^{1,0} + S_{0,1}^{1,0}/2) + R_0^6\rho V^2 C_{2,-1}^{3,-1}/2$, $C_3 = \sigma R_0^2 S_{1,2}^{1,0}/2 + R_0^6\rho V^2(2S_{-1,2}^{3,-1}R_0 + S_{-2,3}^{5,-2} + R_0 S_{-2,3}^{6,-2})/2$, $C_4 = \sigma R_0^2 S_{2,0}^{-1,0}/2$, with $S_{i,j}^{k,l} = R_0^{-l}\int_0^\pi h^l g^i g_\theta^j \sin^k\theta\, d\theta$. Terms proportional to $\eta\eta_\phi^2$ can be neglected since they introduce a factor η_0^3/L^2 which is small compared to η_0^3; i.e., it is third order in ϵ. If Eq. (11) is taken to be a Hamiltonian, $E \to H[\eta]$, then the Hamilton equation for the dynamical variable η, taking the usual form of the Poisson bracket, gives

$$\int_0^{2\pi} \eta_t\, d\phi = \int_0^{2\pi} (2C_2\eta_\phi + 6C_3\eta\eta_\phi - 2C_4\eta_{\phi\phi\phi})\, d\phi\,. \quad (12)$$

Since for the function $\eta(\phi - Vt)$ the left-hand side of Eq. (12) is zero, the integrand on the right-hand side gives the KdV equation. Hence, the energy of the NLD model, in the third order, is interpreted as a Hamiltonian of the KdV equation [7,11]. This is in full agreement with the result finalized by Eq. (7) for an appropriate choice of the parameters and the Cauchy conditions for g, h. The dependence of $E(\alpha_1, \alpha_2)|_{\mathrm{Vol=const}}$, Eq. (11), shows an energy minimum in which the solitary waves are stable [12].

The nonlinear coupling of modes in the cnoidal solution could explain the occurrence of many resonances for the $l = 2$ mode of rotating liquid drops, at a given (higher) angular velocity [13]. The rotating quadrupole shape is close to the soliton limit of the cnoidal wave. On one hand, the existence of many resonances is a consequence of the multivalley profile of the effective potential energy for the KdV equation: $\eta_x^2 = a\eta + b\eta^2 + c\eta^3 + (d\eta^4)$ [11]. The frequency shift predicted by Busse in [13] can be reproduced in the present theory by choosing the solution $h_1 = R_0 \sin\theta/2$. It results in the same additional pressure drop in the form of $V^2\rho R_0^2 \sin^2\theta/2$ as in [13], and hence a similar result. For a roton emerged from a $l = 2$ mode, by calculating the half-width (L_2) and amplitude ($\eta_{\max,2}$) which fit the quadrupole shape, a law for the frequency shift can be given: $\Delta\omega_2/\omega_2 = [1 \pm 4L^2(\alpha_3 - \alpha_2)/3R_0]^{-1} V/\omega_2$, showing a good agreement with the observations of Annamalai *et al.* in [13], i.e., many resonances and nonlinear dependence of the shift on $\Omega = V$. The special damping of the $\lambda = 2$ mode for rotating drops could also be a consequence of the existence of the cnoidal solution. Increasing the velocity V produces a modification of the balance of the coefficients C/D which is equivalent to increasing the dispersion.

The model introduced in this Letter proves that traveling analytic solutions exist as cnoidal waves on the surface of a liquid drop. These traveling deformations (rotons) can range from small oscillations (normal modes), to cnoidal oscillations, and on out to solitary waves. The same approach can be applied to bubbles as well, except that the boundary condition on Σ_2 is replaced by a far-field condition [2,3] (recently important in the context of single bubble sonoluminiscence). Nonlinear phenomena cannot be fully investigated with normal linear tools, e.g., spherical harmonics. Using analytic nonlinear solutions sacrifices the linearity of the space but replaces it with multiscale dynamical behavior, typical for nonlinear systems (solitons, wavelets, compactons [12]). They can be applied to phenomena like cluster formation in nuclei, fragmentation or cold fission, the dynamics of the pellet surface in inertial fusion, stellar models, and so forth.

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