Selection of the Saffman-Taylor Finger Width in the Absence of Surface Tension: An Exact Result

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Using exact time-dependent nonsingular solutions [Mineev-Weinstein and Dawson, Phys. Rev. E **50**, R24 (1994); Dawson and Mineev-Weinstein, Physica (Amsterdam) **73D**, 373 (1994)], we solve the Saffman-Taylor finger selection problem in the absence of surface tension. We show that a generic interface in a Hele-Shaw cell evolves to a nonlinearly stable single uniformly advancing finger occupying one-half of the channel width. This result contradicts the generally accepted belief that surface tension is indispensable for the selection of the $\frac{1}{2}$-width finger.   [S0031-9007(98)05459-3]

The problem of the finger width selection was posed in 1958 by Saffman and Taylor [1] in their study of displacement of oil by water in a Hele-Shaw cell. This cell consists of two parallel glass plates separated by a thin gap occupied by a viscous liquid which is pushed by a less viscous one. This simple device is useful for modeling flows in porous media, the study of which is vitally important for many applications. Flows in uniform porous media and in the Hele-Shaw cell are described by the same Darcy law: $\mathbf{v} = -\nabla p$, where $\mathbf{v}$ is fluid velocity, and p is pressure. For the Hele-Shaw cell, this equation follows from the Stokes equation averaged over the direction perpendicular to the parallel plates.

Saffman and Taylor [1] observed that an almost planar initial oil/water interface becomes unstable and gives rise to many competing fingers which eventually evolve to a single uniformly advancing finger occupying *one-half* of the channel width, if the surface tension is very small. But, as was analytically shown in the same paper [1], *any finger width* is possible. So the selection problem was stated: why does nature choose the width of one-half?

This problem appeared to be universal; i.e., the same selection phenomenon is common for displacement of various viscous liquids by less viscous ones for immiscible incompressible liquids. This problem is related to the problem of pattern selection in nonequilibrium phenomena, which has been of much subsequent interest [2].

When the viscosity of water is negligible compared with oil viscosity, the mathematical formulation of this problem in the absence of surface tension has the form

$$\begin{cases} \nabla^2 p = 0 & \text{(in the oil domain)}, \\ p = -x & \text{if } x \to +\infty \quad \text{(oil pushed to the right)}, \\ p = 0 & \text{(at the oil/water interface)}, \\ \partial_n p = 0 & \text{for } y = \pm\pi \quad \text{(at the channel walls)}, \\ V_n = -\partial_n p & \text{(at the oil/water interface)}, \end{cases} \tag{1}$$

where V_n is the normal velocity of the interface, $\partial_n p$ is the normal component of the pressure gradient, and the channel width is chosen to be 2π in our scaled units.

The solution of the system (1), describing a finger moving in the x direction with velocity $1/\lambda$ and occupying the portion λ of the Hele-Shaw channel width, is [1]

$$x = 2(1 - \lambda)\ln[\cos(y/2\lambda)] + t/\lambda. \tag{2}$$

We parametrize the moving interface $z(t, \phi) = x(t, \phi) + iy(t, \phi)$ at time t by the parameter $\phi \, \epsilon \, [0, 2\pi]$. After the shift $y \to y - \pi\lambda$, Eq. (2) can be rewritten as

$$z(t, \phi) = t/\lambda + (1 - \alpha)i\phi + \alpha \ln[(e^{i\phi} - 1)/2i], \tag{3}$$

where $\lambda = 1 - \alpha/2$. The system (1) can be reduced to what we call the Laplacian growth equation (LGE) for the moving front $z(t, \phi)$ (see [3] and references therein),

$$\text{Im}(\bar{z}_t z_\phi) = 1. \tag{4}$$

Here the bar denotes the complex conjugate, z_t and z_ϕ are partial derivatives, and the map $z(t, \phi)$ is conformal for Im $\phi \leq 0$. One can easily see that $z(t, \phi)$ given by (3) is the traveling-wave solution of the LGE given by (4). The finger width λ is here a free parameter, while experimentally it is always $\frac{1}{2}$. What determines λ?

In [1] Saffman and Taylor proposed that surface tension between the two fluids would solve the selection problem. Since then, it has been widely accepted that *the inclusion of surface tension is the only way to select the most stable finger width*, and much work was done toward solving the selection problem in this way (see books [4] and references therein). While mathematically nontrivial and challenging [4], this activity, especially intensive in the 1980s, was nonetheless successfully completed [4] and summarized in [5]. In short, several groups in 1986–1987 [6] confirmed (using expansion "beyond all orders" and reduction to a nonlinear eigenvalue problem) numerical evidence [7] of the discrete spectrum of λ, decreasing to $1/2$, in the limit of low surface tension. Surface tension was claimed to be responsible for the selection: equating it to zero would make this analysis senseless.

We see two reasons for the absence of attempts to explain the selection without the inclusion of surface

tension. First, because of the absence of analytic time-dependent solutions, all selection studies were focused on *linear stability* analysis of a *steady-state* traveling finger in the presence of surface tension, which is the main physical factor neglected when the continuous family of fingers (2) was derived [1]. The second reason to include surface tension stems from the observation that almost all exact zero surface tension solutions of this problem obtained before 1994 [8] exhibit finite-time singularities (cusps). Because of the belief that these solutions are general and thus capture main features of this problem, it was concluded that to reach long times, it is necessary to include surface tension to eliminate singularities [9].

In 1994, we reported [3] a new class of exact time-dependent solutions of the LGE (4) having the form

$$z(t, \phi) = \tau(t) + i\mu\phi + \sum_{k=1}^{N} \alpha_k \ln[e^{i\phi} - a_k(t)], \quad (5)$$

where $\mu = 1 - \sum_{k=1}^{N} \alpha_k$, $\alpha_k = \text{const}$, and $|a_k| < 1$. With some (quite modest) constraints on $\{\alpha_k\}$, these solutions remain nonsingular and analytic for all times (no cusps) [3]. The time dependence of $a_k(t)$ and $\tau(t)$ is given by

$$\beta_k = z(t, i \ln \bar{a}_k) = \tau - \left(1 - \sum_{l=1}^{N} \alpha_l\right) \ln \bar{a}_k$$

$$+ \sum_{l=1}^{N} \alpha_l \ln\left(\frac{1}{\bar{a}_k} - a_l\right) = \text{const}, \quad (6)$$

$$t + C = \left(1 - \frac{1}{2}\sum_{k=1}^{N} \alpha_k\right)\tau + \frac{1}{2}\sum_{k=1}^{N} \alpha_k \ln(a_k), \quad (7)$$

where $k = 1, 2, \ldots, N$ and C is a constant in time [10]. Equations (6) and (7) follow from the substitution of (5) into (4).

All a_k are located inside the unit circle and, if the same holds for the roots of z_ϕ, then $z(t, \phi)$ is conformal for Im $\phi \leq 0$. We called these solutions N-finger solutions, since they describe the evolution of N fingers. The class of solutions (5) contains all previously known exact solutions [8,11,12], including those which diverge in finite time. The subclass of these solutions without finite-time singularities was also shown to be dense in the space of all analytic curves [13]. *The dynamics of an arbitrary initial interface can be faithfully described within this class.* In addition to possessing these attractive mathematical properties, these solutions are very physical: They describe tip splitting, sidebranching, competition, coarsening, and screening of growing fingers which are observed in all known experiments and simulations.

The following geometrical interpretation of the constants $\{\alpha_k\}$ and $\{\beta_k\}$ [3] is of great help: the complex number $\beta_k - \alpha_k \ln 2$ is the location of the kth stagnation point which the interface does not cross (tips of white grooves in Figs. 1 and 2), but approaches expo-
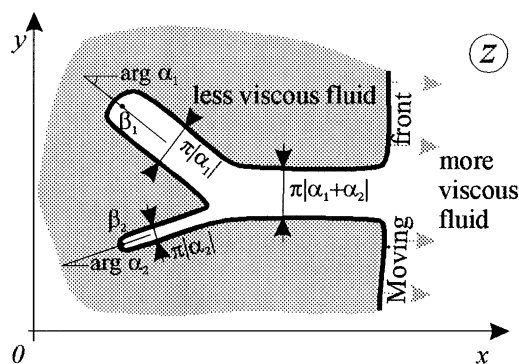

FIG. 1. Geometrical interpretation of the complex constants of motion α_k and β_k; $k = 1, \ldots, N$.

nentially slowly, namely, proportional to $\exp(-\tau/\text{Re }\alpha_k)$. Near the kth stagnation point, a groove with parallel walls originates, with width $\pi|\alpha_k|$ and angle with respect to the horizontal axis $\theta_k = \arg \alpha_k$. In terms of these stagnation points and grooves given by constants $\{\beta_k\}$ and $\{\alpha_k\}$, all the dynamical features mentioned above are especially clear (see Fig. 1). These grooves merge and finally coalesce to a single growing finger in accordance with all known experiments and simulations [1,4]. Formally this means that a generic initial interface given by the N-finger solution (5) necessarily evolves to a single uniformly advancing finger (see [14] for details).

In this paper, we will solve the $\frac{1}{2}$-width selection problem analytically in the absence of surface tension (*while interfacial tension is required to linearly stabilize the finger, as previously shown* [4–6]). It is known [11] that the development of a single finger with a relative width $\lambda = 1 - \alpha/2$ in the long-time limit is described by

$$z(t, \phi) = \tau(t) + \mu i\phi + \alpha \ln[e^{i\phi} - a(t)], \quad (8)$$

where $0 < \alpha < 2$, $0 < a < 1$. For $t \to \infty$, $\tau = 2t/(2 - \alpha)$ and $a(t) = 1 - c \exp[-2t/\alpha(2 - \alpha)]$, where c and α are constants in time. Choosing the width of the Hele-Shaw cell to be 2π, we have $z(t, 2\pi) - z(t, 0) = 2\pi i$, because of the periodic boundary conditions. Calculating $z(t, 2\pi) - z(t, 0)$ from (8) and using the fact that $|a| < 1$, we obtain that $2\pi i = 2\pi i\mu + 2\pi i\alpha = 2\pi i(\mu + \alpha)$, or finally

$$\mu + \alpha = 1. \quad (9)$$

Then we note that the second term in the right-hand side of (8), namely, $\mu i\phi$, is the limiting value of the logarithm with a logarithmic pole ϵ located at zero,

$$\mu i\phi = \mu \lim_{\epsilon \to 0} \ln(e^{i\phi} - \epsilon). \quad (10)$$

Let us perturb the interface (8), corresponding to $\epsilon = 0$, by the initially small nonzero ϵ, $0 < \epsilon \ll 1$. The perturbed interface will have the form

$$z(t, \phi) = \tau + \mu \ln(e^{i\phi} - \epsilon) + \alpha \ln(e^{i\phi} - a). \quad (11)$$

Now one can easily see that the value $\epsilon = 0$ [and thus the finger described by (8) with $\mu \neq 0$] is unstable. The point is that Eq. (11) is exactly the $N = 2$ case of the N-finger solution (5) of the LGE (4). As one can see from (6) and (7), $a(t)$ and $\epsilon(t)$ merge at unity when $t \to \infty$,

$$a = 1 - l_1 e^{-\tau}, \qquad \epsilon = 1 - l_2 e^{-\tau}, \qquad (12)$$

where the constants l_1 and l_2 are determined by

$$\begin{cases} \beta_1 = \mu \ln(l_1 + l_2) + \alpha \ln(2l_1), \\ \beta_2 = \mu \ln(2l_2) + \alpha \ln(l_1 + l_2). \end{cases} \qquad (13)$$

In view of (12), we substitute 1 for both a and ϵ with the accuracy $O(e^{-\tau})$, and thus obtain from (11), for $t \to \infty$,

$$z(t, \phi) = \tau + (\mu + \alpha) \ln(e^{i\phi} - 1). \qquad (14)$$

Let us interpret the result (14): because of the instability of the initial finger (8) with width $\lambda = 1 - \alpha/2$, the new finger described by (14) has been formed. Its width is

$$\lambda^{\text{new}} = 1 - (\mu + \alpha)/2 = 1/2, \qquad (15)$$

in accordance with the condition (9) that $\mu + \alpha = 1$.

Let us perturb the finger (8) in a more general way than we did in (11). We note that

$$\mu i \phi = \lim_{\text{all } \epsilon_k \to 0} \sum_{k=1}^{N} \delta_k \ln(e^{i\phi} - \epsilon_k)$$

if $\sum_{k=1}^{N} \delta_k = \mu$. Choosing all ϵ_k to be nonzero, we rewrite the finger (8) in a perturbed way as

$$z(t, \phi) = \tau + \sum_{k=1}^{N} \delta_k \ln(e^{i\phi} - \epsilon_k) + \alpha \ln(e^{i\phi} - a). \qquad (16)$$

Equation (16) is the $(N + 1)$-finger solution (5) of the LGE (4) with dynamical conditions (6) and (7). Because of the density of the subclass of smooth solutions given by (5) we conclude that (16) describes a general perturbation of the finger (8), if N is large enough. As follows from (6) and (7), generally all logarithmic poles in the absence of finite-time singularities *merge* [similarly to (12) for $N = 2$] in the long-time limit to 1 with exponential accuracy $O(e^{-\tau})$ [14], where

$$t = \tau \left[1 - \frac{1}{2} \left(\sum_{k=1}^{N} \delta_k + \alpha \right) \right]. \qquad (17)$$

Because of this merging near the unit circle we substitute 1 for all $\epsilon_k(t)$ and $a(t)$ in (16) when $t \to \infty$ and obtain

$$z(t, \phi) = \tau + \left(\sum_{k=1}^{N} \delta_k + \alpha \right) \ln(e^{i\phi} - 1). \qquad (18)$$

This formula describes the single finger formed from (8) under the perturbation (16). Its width is

$$\lambda^{\text{new}} = 1 - \left(\sum_{k=1}^{N} \delta_k + \alpha \right) \Big/ 2, \qquad (19)$$

which is exactly one-half since $\sum_{k=1}^{N} \delta_k = \mu$ (see above) and $\mu + \alpha = 1$ by an argument analogous to (9), so that

$$\lambda^{\text{new}} = 1 - (\mu + \alpha)/2 = 1/2. \qquad (20)$$

Both (20) and (15) indicate that, for obtaining instability of non-$\frac{1}{2}$-width finger and formation of the $\frac{1}{2}$-width finger in a long-time limit, surface tension is not needed.

Now we start from an arbitrary initial interface in terms of nonsingular solutions (5) (Re $\alpha_k > 0$). Because of the coalescence of all initially nonzero poles a_k at the unit circle, the long-time limit of (5) is given by a finger (3) with a width of $(2 - \sum_{k=1}^{N} \alpha_k)/2$. The N-finger solution (5) is the limit of $(N + k)$-finger solution expressed by the same equation, but without the term $\mu i \phi$. This limit corresponds to equating k of the poles to zero, and this value of zero can easily be shown (in the same way as above) to be unstable for all of these k poles. Because of the density of these solutions in the class of all analytic curves [13], this $(N + k)$-finger solution can be arbitrarily close to any analytic interface. Again, all $a_k(t)$ merge to 1 in the limit $t \to \infty$ [as stated earlier and proven in [14]]. Thus we have in the long-time limit

$$z(t, \phi) = \tau + \left(\sum_{k=1}^{N+k} \alpha_k \right) \ln(e^{i\phi} - 1)$$
$$= \tau + \ln(e^{i\phi} - 1).$$

Here we used $\sum_{k=1}^{N+k} \alpha_k = 1$, since $2\pi i = z(t, 2\pi) - z(t, 0)$. So, we have demonstrated that initial interfaces evolving to the non-$\frac{1}{2}$-width finger are unstable with respect to formation of the $\frac{1}{2}$-width finger, which thus is shown to be the only attractor for all generic moving fronts in the Hele-Shaw cell represented by (5). (Solutions with several parallel fingers forming in asymptotics can also be easily shown to be unstable.) The dynamics of the transition from an arbitrary interface to the $\frac{1}{2}$-width finger is exactly described by the set of transcendental equations (6) and (7) which involve only elementary functions. This selection of one-half is in agreement with known experiments and simulations in the limit of low surface tension.

Now we will extend these results obtained for periodic boundary conditions to the more physical "no-flux" boundary conditions (no flow across the lateral boundaries of the channel). This requires that the moving interface orthogonally intersects the walls of the Hele-Shaw cell. However, unlike the case of periodic boundary conditions, the end points at the two boundaries do not necessarily have the same horizontal coordinate. This is also a periodic problem where the period equals twice the width of the Hele-Shaw cell. The analysis is the same as before, but now only half of the strip should be considered as the physical Hele-Shaw channel, while the second half is the unphysical mirror image (see Fig. 2).

To be brief we will perform the analysis for $N = 2$, but one can trivially extend it to an arbitrary N. The easily obtainable extension of (8) for the development of a single
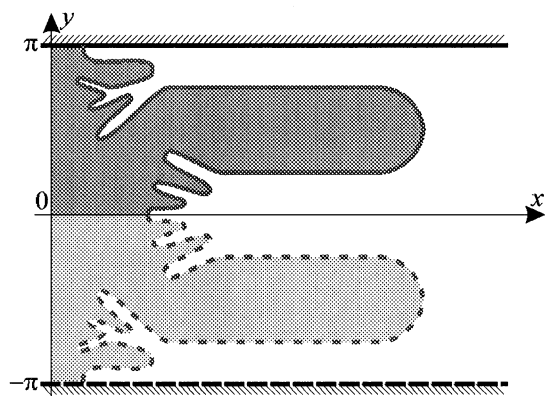
FIG. 2.   Only half of the periodic cell should be considered as the physical Hele-Shaw channel, while the second half is the unphysical mirror image.

finger with the width $\lambda = 1 - (\alpha_1 + \alpha_2)/2$ is

$$z(t,\phi) = \tau(t) + \mu i\phi + \alpha_1 \ln[e^{i\phi} - a_1(t)]$$
$$+ \alpha_2 \ln[e^{i\phi} + a_2(t)], \qquad (21)$$

where $0 < a_{1,2}(t) < 1$. The generalization of (9) is

$$\mu = 1 - (\alpha_1 + \alpha_2). \qquad (22)$$

We note that

$$\mu i\phi = (\tfrac{1}{2} - \alpha_1 + \delta) \lim_{\epsilon_1 \to 0+} \ln(e^{i\phi} - \epsilon_1)$$
$$+ (\tfrac{1}{2} - \alpha_2 - \delta) \lim_{\epsilon_2 \to 0+} \ln(e^{i\phi} + \epsilon_2). \qquad (23)$$

Substituting this into (21) and allowing $\epsilon_{1,2}$ to be functions of time, we note that $z(t,\phi)$ is the $N = 4$ solution (5) of Eq. (4), where δ and $\epsilon_{1,2}$ must be real. If $|\delta| < 1/2$, all four logarithmic singularities $a_{1,2}(t)$ and $\epsilon_{1,2}(t)$ merge to 1. So we have in the long-time limit

$$z(t,\phi) = \tau(t) + (\tfrac{1}{2} + \delta)\ln(e^{i\phi} - 1)$$
$$+ (\tfrac{1}{2} - \delta)\ln(e^{i\phi} + 1). \qquad (24)$$

This describes a finger moving between two grooves with widths $\pi(1 + 2\delta)/2$ and $\pi(1 - 2\delta)/2$, respectively (see the geometrical interpretation above). Thus the portion of the channel width occupied by the moving finger is

$$\lambda^{\text{new}} = \frac{2\pi - \pi(1 + 2\delta)/2 - \pi(1 - 2\delta)/2}{2\pi} = \frac{1}{2}, \qquad (25)$$

as before. So, for the no-flux boundary conditions, we have obtained the $\frac{1}{2}$-width finger as expected. In experiments, the finger in the long-time limit is centralized in the sense that axes of symmetry of the finger and Hele-Shaw cell coincide. This corresponds to the condition $\delta = 0$, so both grooves are of equal width.

The finger with a width of one-half is *nonlinearly stable with respect to generic perturbation of logarithmic type*

(5). Namely, the shape of the finger can be destroyed at the initial (linear) stage, but eventually the $\frac{1}{2}$-width finger will be restored, because of the coalescence described above. (Of course, the $\frac{1}{2}$-width finger is *linearly unstable without interfacial tension in accordance with previous studies* [4–6].)

Regarding surface tension, we think that, while mathematically still singular (because a small number multiplies the highest derivative), physically surface tension is a *regular* perturbation for this problem (unless very high curvatures exist which surface tension suppresses).

In conclusion, we have analytically solved the finger selection problem in the absence of surface tension. By using the nonsingular exact solutions of the LGE (4), we have demonstrated that the $\frac{1}{2}$-width finger is the only attractor for all generic moving fronts in a Hele-Shaw cell in the long-time limit.

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[10] Particular examples of this class, limited, however, to a specific symmetry, were known earlier [11,12]. The solution (5) also has the symmetry such that the sum in the right-hand side of (5) contains both terms with α_k and a_k and with $\bar{a}_k$ and $\bar{a}_k$ (unless both α_k and a_k are real). It is related to no-flux boundary conditions (see discussion and analysis in this paper beginning at "Now we will extend these results..." and in Ref. [14]).

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