Fractal Conductance Fluctuations in a Soft-Wall Stadium and a Sinai Billiard

A. S. Sachrajda,[1] R. Ketzmerick,[2] C. Gould,[1,3] Y. Feng,[1] P. J. Kelly,[1] A. Delage,[1] and Z. Wasilewski[1]
[1] *Institute for Microstructural Sciences, National Research Council of Canada, Ottawa, Canada K1A0R6*
[2] *Max-Planck-Institut für Strömungsforschung und Institut für Nichtlineare Dynamik der, Universität Göttingen, Bunsenstrasse 10, D-37073 Göttingen, Germany*
[3] *Département de Physique and CRPS, Université de Sherbrooke, Sherbrooke, Canada J1K 2R1*
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Conductance fluctuations have been studied in a soft-wall stadium and a Sinai billiard defined by electrostatic gates on a high mobility semiconductor heterojunction. These reproducible magnetoconductance fluctuations are found to be fractal, confirming recent theoretical predictions of quantum signatures in classically mixed (regular and chaotic) systems. The fractal character of the fluctuations provides direct evidence for a hierarchical phase space structure at the boundary between regular and chaotic motion. [S0031-9007(98)05438-6]

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Ballistic geometries defined in two-dimensional electron gases (2DEGs) provide excellent systems for studying quantum chaotic phenomena. In magnetic fields these systems yield reproducible conductance fluctuations and weak localization effects analogous to the universal conductance fluctuations and weak localization effects studied in disordered conductors. In the ballistic case the phase coherent phenomena reflect the device geometry; not the random positions of impurities. By studying these effects in both chaotic and nonchaotic geometries, valuable insight has been obtained on how quantum information can be retrieved from classical chaotic dynamics. To date, in virtually all of the systems studied both experimentally and theoretically, a purely chaotic system has been assumed in which classical trajectories probe the phase space in a purely ergodic fashion [1–3] before they exit. Escape from the ballistic cavity in such systems usually occurs exponentially fast.

Real billiards, however, are typically not fully chaotic. They have a *mixed* phase space, i.e., they contain both chaotic and regular regions. Naively, one might expect that they are simply combinations of independent fully chaotic and regular regions. But this is not the case. Mixed systems possess an important property which has drastic consequences for conductance fluctuations: The chaotic part of phase space obeys a power law escape probability [4], in contrast to the much faster exponential decay of fully chaotic systems. The power law originates from chaotic trajectories which are "trapped" close to an infinite hierarchy of regular regions at the boundary between the regular and chaotic motion (see Fig. 3 below). This trapping is believed to be a consequence of Cantori, which act as partial barriers for transport and which surround the regular regions at all levels of the hierarchy [5,6]. The phase of these long trapped trajectories is extremely sensitive to any externally changed parameter. In a magnetic field, for example, they acquire a phase factor $\exp(2\pi i AB/\phi_0)$, which depends on the number AB/ϕ_0 of magnetic flux quanta $\phi_0 = h/e$ enclosed by the trajectory (A is the accumulated enclosed area). The larger the value of A, the smaller the change in magnetic field required to modify the phase. Therefore, in these systems, one expects to observe conductance fluctuations with much finer field scales [7,8]. Semiclassical calculations similar to those in Refs. [1,2,7] have demonstrated that a power law

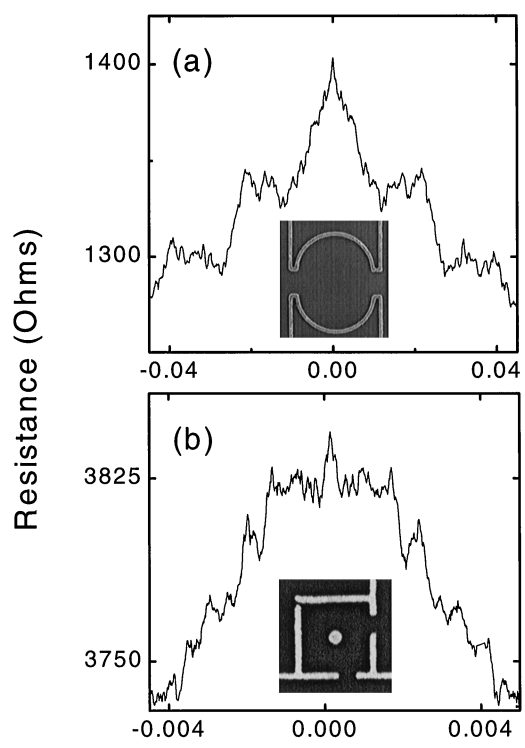

FIG. 1. Resistance versus magnetic field (a) for the open stadium at a gate voltage of -1.9 V after illumination at 50 mK and (b) for the Sinai Billiard at 50 mK. Fluctuations on both large and small scales can be seen for both devices. The two insets are scanning electron micrographs of devices similar to the ones used. In the actual Sinai billiard used for these measurements an insulator bridge connection (not shown) was used to make contact to the center gate of the Sinai device.

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distribution of enclosed areas larger than A,

$$P(A) \sim A^{-\gamma}, \tag{1}$$

leads to *fractal* conductance fluctuations [8]. Under these conditions, the line conductance versus magnetic field has the same properties as fractional Brownian motion [9], which is self-affine in a statistical sense and is described by a fractal dimension D given by

$$D = 2 - \gamma/2. \tag{2}$$

Thus, the exponent γ, which describes a property of the classical phase space, is related to the fractal dimension D of a quantum coherent measurement. Similarly, the variance of conductance increments scale with small magnetic field increments as $\langle(\Delta G)^2\rangle \sim (\Delta B)^\gamma$. These results hold [8] for $\gamma \leq 2$. Since $\gamma > 1$ must hold from more general arguments [10], this restricts the fractal dimension to lie between 1 ($\gamma = 2$) and 1.5 ($\gamma = 1$). While the value of the classical exponent γ is nonuniversal and is found numerically to be sensitive to details of the geometry and the confining potential [8], the occurrence of power law distributions is universal [4]. It is not feasible to test the existence of fractal conductance fluctuations by numerical quantum calculations, since a large number of modes are needed to adequately probe the hierarchical classical phase space. Indeed, because the semiclassical calculations of Ref. [8] are based on many assumptions and approximations, one may even speculate if fractal fluctuations exist at all.

The first experimental evidence for fractal behavior was recently found in gold nanowires [11]. The variance analysis of the fluctuations showed a power law behavior over one order of magnitude in magnetic field. While suggestive, this experiment suffered from two limitations: (i) It is well known that almost any smooth function can be satisfactorily fitted by a power law for just one order of magnitude, and (ii) for the rectangular geometry of these gold nanowires one does not expect a mixed phase space, unless fortuitous fluctuations in wire width occasionally create such regions [12]. Herein, we report the first observation of fractal conductance fluctuations over *2 orders* of magnitude in magnetic field and in genuine mixed phase space geometries. The two classic models of fully chaotic systems, namely, the stadium (two half-circles connected by straight lines) and the Sinai billiard (a square with a circular disk at its center) are defined by electrostatic gates on a high mobility semiconductor heterojunction. These systems are no longer fully chaotic due to soft-wall potentials, but have a mixed phase space. This can be easily verified numerically (see Fig. 3 below). We find fractal conductance fluctuations as a function of magnetic field for large ranges of applied gate voltage (corresponding effectively to different billiard sizes and potential forms). The existence of this new quantum signature of classically mixed systems is therefore conclusively confirmed. In addition, by choosing the stadium geometry, we have shown that the famous experiment by Marcus *et al.* [3] on the stadium, which was performed to study the properties of fully chaotic systems, gives qualitatively different results [13] when carried out on today's high mobility samples.

The two devices are shown as insets in Fig. 1. The stadium had a lithographic radius of 1.1 μm. It was defined using metallic gates on a high mobility AlGaAs/GaAs wafer [with a mobility of 2.2×10^6 and 4.3×10^6 cm^2/V s and density of 1.7×10^{11} and 3.3×10^{11} cm^{-2} before and after illumination with a red light-emitting diode (LED)]. The 2DEG was 95 nm below the surface. The device leads were made unusually wide (0.7 μm). This had the effect of allowing most trajectories to rapidly exit the stadium, with the exception of those trajectories which were trapped near the hierarchical phase space structure at the boundary between regular and chaotic motion. This feature and the high mobility wafer used made this an optimal device for the observation of fractal conductance fluctuations. The very high mobility wafer was necessary to achieve the required long phase coherence length l_φ. The sub-mT features in the conductance fluctuations and the narrow weak localization peaks (< 150 μT) confirmed that l_φ was indeed many times the perimeter of the stadium and Sinai billiard. Lithographic and fabrication details of the Sinai billard have been published elsewhere [14]. The measurements were made on a dilution refrigerator using standard low power ac techniques at 50 mK.

Measurements at higher temperatures ($\sim$4 K) revealed small features in the magnetoresistance related to the classical focusing of trajectories. As the temperature was lowered, both conductance fluctuations and a weak localization peak (about $B = 0$) developed. In this paper, we analyze the fluctuations which occur at magnetic fields for which the cyclotron diameter is larger than the stadium diameter. Features related to noise can be eliminated by comparing the $\pm B$ traces. In Ref. [14] it was observed for a soft-wall Sinai billiard that, by rescaling magnetic field and resistance, one finds quite similar sequences of maxima and minima. This *nonstatistical* self-affinity remains unexplained so far and does not occur for the stadium, suggesting that this feature is characteristic of the Sinai billiard geometry. We observe in contrast conductance fluctuations on very different scales resembling the *statistical* self-affinity of fractional Brownian motion for both geometries and for large ranges of gate voltage.

Experimental traces for both geometries are shown in Fig. 1. The results of a fractal analysis on these curves, shown in Fig. 2, gives fractal dimensions of $D = 1.25$ and $D = 1.30$. The fluctuations are found to obey power law scaling for over two orders of magnitude in magnetic field. In all observable fractals this scaling behavior is bound by upper and lower cutoffs. In our case the upper cutoff, in magnetic field, is determined by the smallest area for which the power law distribution holds (which is close to the range of magnetic fields experimentally studied). The lower cutoff is determined by the minimum of two time scales: (i) The finite phase coherence time and

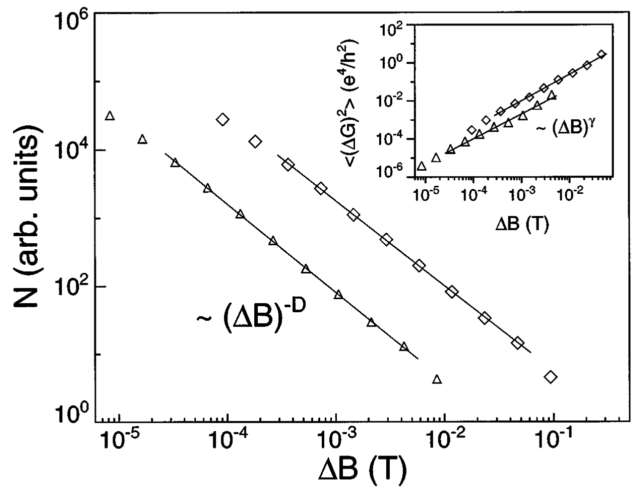

FIG. 2. Fractal analysis for the stadium (diamonds) and Sinai device (triangles) data shown in Fig. 1 using a modified box-counting algorithm. The number of boxes N follows power laws $(\Delta B)^{-D}$ for 2 orders of magnitude in magnetic field scale, giving fractal dimensions $D = 1.25$ and $D = 1.30$, respectively. The inset shows the variance analysis giving $\gamma = 1.38$ and $\gamma = 1.30$, respectively, in reasonable agreement with Eq. (2).

(ii) the Heisenberg time $t_H = \hbar/\Delta E$, when the average level spacing ΔE of the closed device is resolved and the semiclassical approximation becomes unreliable. The power law area distribution for trajectories staying in the device longer than any of these times will not lead to fractal fluctuations. The fractal dimension observed between these cutoffs is determined by using a refined version of the box-counting algorithm. In the standard box-counting algorithm, one puts a grid of square boxes of size $L \times L$ on the data (conductance versus magnetic field) and counts the number $N(L)$ of boxes through which the curve passes. Its dependence on box size L, $N(L) \sim L^{-D}$, defines the fractal dimension D. In this standard analysis the relative scale of conductance and magnetic field is arbitrary so that for a finite data set the resulting fractal dimension depends on the aspect ratio of the plot. To overcome this limitation we applied a modified version of the algorithm: This divides the magnetic field range in length intervals ΔB and determines $N(\Delta B)$ as the difference of maximum and minimum conductance in each interval, summed over all intervals, and divided by ΔB. This corresponds in the standard algorithm to taking rectangular boxes with infinitely small size in the conductance direction. The inset of Fig. 2 shows the analysis of the variance, which gives $\gamma = 1.38$ and $\gamma = 1.30$, for stadium and Sinai billiard, respectively, confirming the above fractal analysis.

We now show that the stadium, a classic model for fully chaotic systems, has a mixed classical phase space when achieved electrostatically due to the soft-wall potential experienced by the electrons. We model this potential by a depletion region of 100 nm at the edge of the device followed by a 400 nm parabolic region, and a flat potential beyond that [15]. The mixed phase space is revealed in a Poincaré section analysis (Fig. 3). Its hierarchical structure gives rise to long trapped trajectories as well as a power law area distribution. The effective Planck's constant $\hbar_{\rm eff} = \hbar/\int p\,dq = (L\sqrt{2\pi n})^{-1}$ for a trajectory along the circumference of the stadium is $\hbar_{\rm eff} = 8.6 \times 10^{-4}$. All of this confirms the applicability of the results of Ref. [8] to the present experiment. Similar conclusions apply to the soft-wall Sinai billiard.

Direct simulation is the only technique available to predict the exact value of the classical power law exponent and the corresponding fractal dimension for a particular device. The values of these exponents depend critically

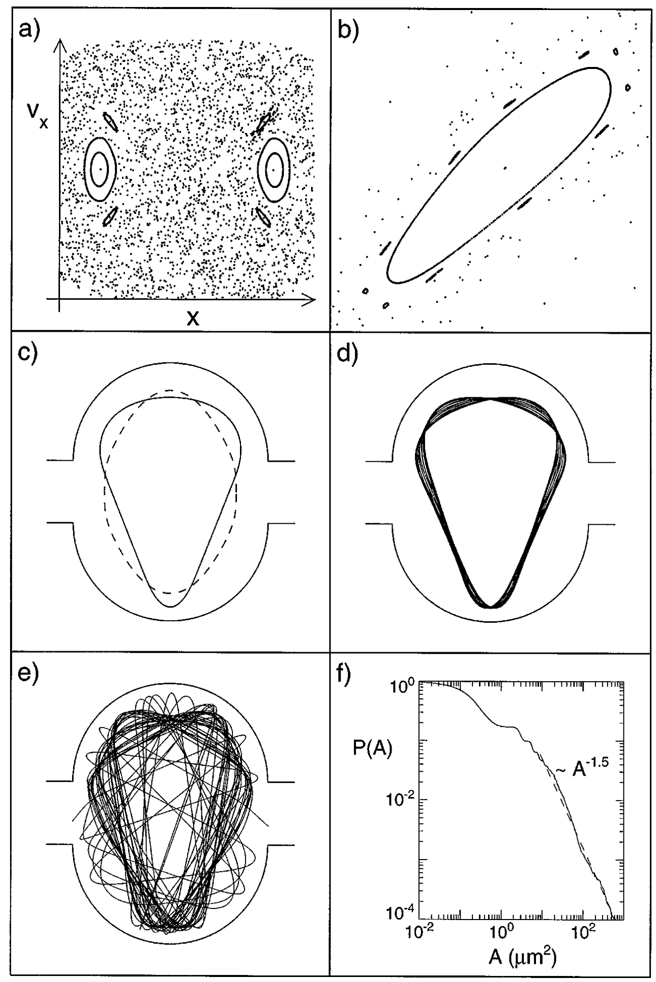

FIG. 3. (a) Poincaré surface of section of the stadium of Fig. 1 (with 0.1 μm depletion length and a 0.4 μm wide parabolic wall). It shows v_x versus x at every trajectory intersection with the horizontal symmetry line whenever $v_y > 0$ holds, for one chaotic and six regular trajectories. (b) Enlargement of the small island in (a) showing higher order islands. (c) Periodic trajectories corresponding to the center of the large island in (a) (dashed line) and the center of the island in (b) (solid line). (d) Quasiperiodic trajectory corresponding to the island in (b). (e) Chaotic trajectory being trapped in the stadium close to the regular trajectory of (d). (f) Integrated area distribution displaying a power law for large enclosed areas.

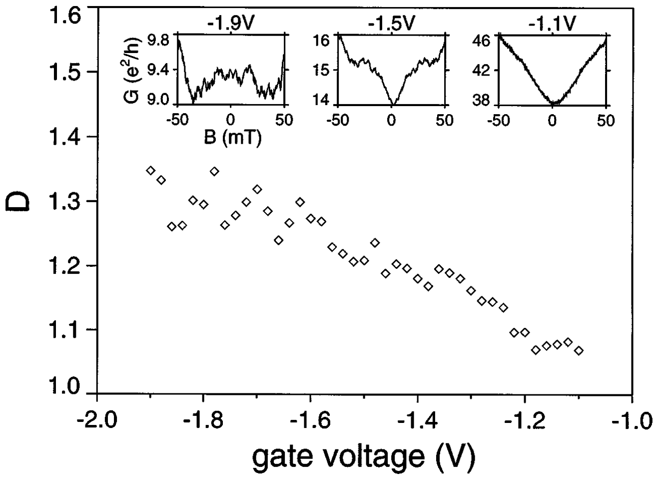

FIG. 4. Fractal dimension of conductance fluctuations versus gate voltage for the stadium before illumination. Error bars of the fractal analysis are typically ±0.05. The insets show data for three gate voltages.

on the exact form of the potential [8]. But even if the electrostatic potential due to the gates is known, it would be modified in a real device by the presence of disorder. In high-mobility samples, however, the disorder potential is weak and smooth and so does not change the qualitative character of the classical phase space, i.e., it remains a mixed phase space. The exponent γ of the power law area distribution of trapped chaotic trajectories on the other hand, might well be changed by the specific disorder configuration inside the cavity. Thus, one cannot hope for a quantitative comparison of the exponent γ deduced from classical simulations without disorder with the fractal dimension D of the conductance fluctuations of a real device containing disorder. Disorder outside the cavity may, in principle, also change the conductance fluctuations, but it cannot influence the trapping of chaotic trajectories inside the cavity and thus it will have no significance for the observed fractal dimension. As a function of the applied gate voltage, the exponents of the classical power law distribution may fluctuate for large times [16], but for times smaller than the phase coherence time the power law exponents have been found to be much more stable [8]. This feature is also reproduced in our experiment. Figure 4 reveals that there exists a monotonic relationship between the fractal dimension and the gate voltage applied to the stadium device. The fractal dimension tends towards 1 as the confinement is reduced.

We have reported the first observation of fractal conductance fluctuations in semiconductor billiard devices [17]. The observation confirms a recent theoretical prediction that conductance fluctuations in mixed phase space systems are statistical self-affine and can be described by a fractal dimension. The origin of the behavior is a power law distribution of areas enclosed by chaotic trajectories, which results from the hierarchical structure of phase space at the boundary of regular and chaotic motion. We have shown that in real devices the classification into "chaotic" and "regular" geometries is incomplete at best. Our findings confirm the important role that soft-wall potentials play in nanostructures. We have also observed a dependence of the fractal dimension of conductance fluctuations on the device parameters.

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