Building Blocks of Spatiotemporal Intermittency

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For the one-dimensional complex Ginzburg-Landau equation (CGLE) we obtain, by a shooting algorithm, a family of uniformly propagating hole solutions which differ from the well-known Nozaki-Bekki holes. These holes occur in many regimes of the CGLE, most prominently in the regime known as spatiotemporal intermittency. A stability analysis reveals that these holes have one unstable core mode, and we discuss the consequence of this for the intermittent states. [S0031-9007(98)05485-4]

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A proper understanding of spatiotemporal chaos, i.e., deterministic chaos occurring in extended systems that are driven from equilibrium, is lacking. Since the number of effective degrees of freedom diverges with the system size, most of the tools developed for low-dimensional systems are inapplicable. Moreover, these tools do not provide a proper framework to describe the spatial organization of extended chaos. In many cases, the dynamical states appear to be built up from local, almost particlelike objects with well-defined dynamics and interactions [Figs. 1(b)–1(d)]. A description of spatiotemporal chaos in terms of these structures is therefore desirable [1].

In this Letter we will investigate local structures that appear mainly in the *spatiotemporal intermittent* regime of the 1D complex Ginzburg-Landau equation (CGLE):

$$A_t = A + (1 + ic_1)A_{xx} - (1 - ic_3)|A|^2A\,. \quad (1)$$

This amplitude equation describes pattern formation near a Hopf bifurcation and has been applied to describe patterns occurring in, e.g., fluid convection, Faraday waves, optical systems, chemical oscillations, and turbulent flow past a wake [2]. As a function of the coefficients c_1 and c_3, which are determined by the underlying physical problem, behavior ranging from completely regular to strongly chaotic has been found [3,4] [Fig. 1(a)].

In the spatiotemporal intermittent regime [4,5], a plane wave attractor coexists with a chaotic attractor; most initial conditions evolve to the latter [Figs. 1(b)–1(d)]. The typical states consist of patches of plane waves, separated by various "holes," i.e., local structures characterized by a depression of $|A|$ [4]. Similar intermittent states have been reported for the damped Kuramoto-Sivashinsky equation, Rayleigh-Bénard convection, the printers instability, and film draining experiments [6–9]. It has been suggested that spatiotemporal intermittency should occur generally in the transition route from laminar to chaotic states, and the phenomenology suggests a relation to directed percolation [6,10].

The local structures in the intermittent regime can be divided into two groups, depending on the wave numbers q_l and q_r of the asymptotic waves they connect. The quasistationary structures in Figs. 1(c) and 1(d) have $q_l \neq q_r$ and are related to the intensively studied Nozaki-Bekki holes [4,12]. However, many local structures have velocities and asymptotic wave numbers that are incompatible with the Nozaki-Bekki holes [4]. For example, the holes shown in Fig. 1(b) all have $q_l \approx q_r \approx 0$. In the following we shall characterize these holes and their dynamical properties, and discuss their relevance for the chaotic states of the CGLE.

In Fig. 2, $|A|$, the complex phase [$\arg(A)$] and the local wave number $q := \partial_x \arg(A)$ for the left-lower part of Fig. 1(b) are shown. The wave numbers of the laminar patches are quite close to zero, while the cores of the local structures are characterized by a sharp phase gradient (peak in q) and a dip of $|A|$. The holes propagate with a speed of 0.95 ± 0.1, and either their phase-gradient

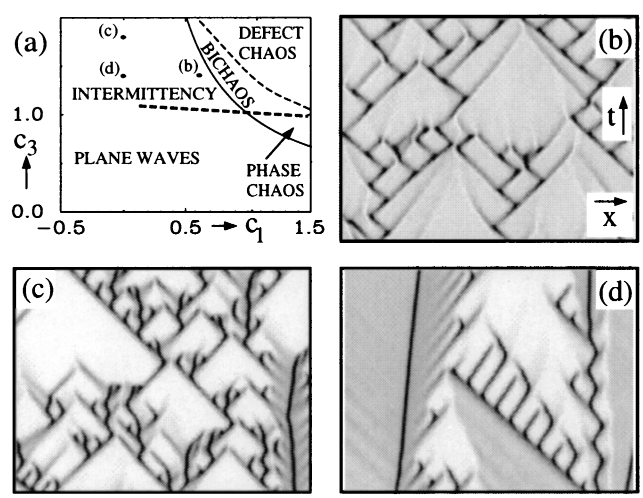

FIG. 1. (a) "Phase diagram" [3,4] of the CGLE. For small c_1 and c_3 all initial conditions evolve to plane waves. In the intermittent regime, a plane wave attractor and a chaotic attractor coexist. Beyond the full curve $c_1c_3 = 1$, all plane waves are linearly unstable and all states are spatiotemporal chaotic. At a zero of A the complex phase is undefined and phase slips occur (see Fig. 2); the chaotic state is then called defect chaos. When A has no zeros we speak of phase chaos. In the bichaotic regime, a defect- and phase-chaotic attractor coexist [4,11]. (b)–(d) Space-time plots (over a range of 200×150) of $|A|$ (black corresponds to $|A| = 0$) showing chaotic states in the spatiotemporal intermittent regime, for coefficients $(c_1, c_3) = (0.6, 1.4)$ (b), $(0, 1.8)$ (c), and $(0, 1.4)$ (d).

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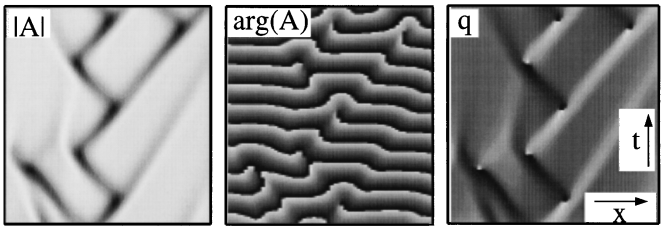

FIG. 2. Space-time (60 × 50) plots of the left-lower part of Fig. 1(b), showing $|A|$, arg $|A|$, and $q := \partial_x \arg(A)$ in detail.

spreads out and the hole decays or the phase gradient steepens and the hole evolves to a phase slip. As a first step in describing these *local structures* which have a slowly evolving velocity and spatial structure, we will study the *coherent* structures, i.e., structures with *fixed* spatial structure and velocity. Unless noted otherwise we take $c_1 = 0.6$ and $c_3 = 1.4$.

For the 1D CGLE, coherent structures have been described in a simple framework [13]. By substituting an *ansatz* [14] for a uniformly propagating solution of the form $A(x,t) = e^{-i\omega t}\hat{A}(\xi)$ into the CGLE (1) ($\xi := x - vt$), we obtain a set of coupled first order ordinary differential equations (ODE's) ode

$$\partial_\xi a = \kappa a\,, \tag{2a}$$

$$\partial_\xi z = -z^2 + \frac{1}{1 + ic_1}\left[-1 - i\omega + (1 - ic_3)a^2 - vz\right], \tag{2b}$$

where $a := |\hat{A}|$ and where the complex quantity z is defined as $\partial_\xi \ln(A) =: \kappa + iq$. Equation (2b) is equivalent to two real-valued equations, so (2) can be seen as a 3D real-valued dynamical system [13]. Plane waves correspond to fixed points of (2), and the hole solutions we are interested in correspond to orbits connecting these fixed points. A rather complete study of the heteroclinic orbits which describe, for instance, the Nozaki-Bekki holes ($q_l \neq q_r$) has been made [13]. Here we are interested in local structures that have $q_l = q_r$, i.e., *homoclinic* orbits of the ODE's (2).

In general, the ODE's (2) have ω and v as free parameters, but since the wave number in the laminar patches is approximately zero, we demand $q_l = q_r = 0$, which fixes $\omega = -c_3$. The fixed point at $(a,z) = (1,0)$ corresponds to the $q = 0$ plane waves and has a 1D outgoing manifold and a 2D spiraling ingoing manifold [Fig. 3(a)]. To create a homoclinic orbit, we have to connect these manifolds. This amounts to satisfying a single condition and, since we have one free parameter (v), we can expect a discrete set of homoclinic orbits for $q_l = q_r = 0$. Performing a simple numerical integration of (2) and adjusting the free parameter v, we obtain a homoclinic orbit for $v \approx 0.916$. The corresponding coherent structures [Fig. 3(b)] will be referred to as homoclinic holes. For $q_l = q_r \neq 0$, i.e., $\omega \neq -c_3$, one can obtain similar homoclinic orbits, so in fact there exists a one-parameter family of homoclinic holes.

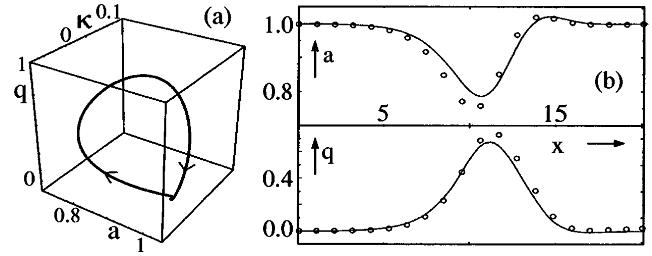

FIG. 3. (a) The homoclinic orbit of the ODE's (2) in a, q, κ space. (b) The amplitude and wave number profile of the corresponding coherent structures (curves). The bump in a to the right of the core corresponds to the spiraling motion on the incoming manifold of the fixed point $(a, q, \kappa) = (1, 0, 0)$. The circles correspond to a local structure obtained from simulations of the CGLE in the spatiotemporal intermittent regime.

In Fig. 3(b) the homoclinic holes are compared to the local structures in the intermittent regime. The slight deviation between this particular local structure and the hole is due mainly to the fact that the local structure has a slowly evolving shape. The longer the lifetime of the dynamical holes, the better the fit is to the homoclinic holes. For nearby values of c_1 and c_3, one finds similar correspondences between the $q = 0$ homoclinic holes and the local structures.

It is instructive to compare the homoclinic holes with the continuous family of Nozaki-Bekki holes [12]. The Nozaki-Bekki holes contradict naive counting arguments [13]. However, under arbitrarily small perturbations $\sim\delta|A|^4 A$, only a single Nozaki-Bekki hole survives, in agreement with counting arguments [15]. In contrast, the homoclinic holes satisfy the counting arguments, and their existence is insensitive to perturbations. One can verify that, for our choice of coefficients, a Nozaki-Bekki hole with $q_l = 0$ has $q_r \approx 0.837$ and velocity 1.673, completely different from the local structures here. Furthermore, in the limit where $q_l = q_r$, the width of the Nozaki-Bekki holes diverges. We conclude that there are two distinct types of hole solutions: heteroclinic Nozaki-Bekki holes and homoclinic holes [16,19].

So what is the interplay between the homoclinic holes and the Nozaki-Bekki holes? In fact, the intermittent state consists of many qualitatively different states [4]. For $c_1 < 0$ we get mixed states, where both types of holes play a role [Fig. 1(c)] [4], and as the Nozaki-Bekki holes are sources for waves with $q \neq 0$, we obtain grain boundaries between $q \neq 0$ and $q \approx 0$ waves [Figs. 1(c) and 1(d)]. Moving to more negative c_1, "glassy" states consisting of Nozaki-Bekki holes only were found [4]. To get intermittency, i.e., interplay between laminar and chaotic patches, the homoclinic holes seem to be essential. For $c_1 > 0$ they are the dominant local structures, and their importance extends into the bichaotic and defect-chaotic regime [4].

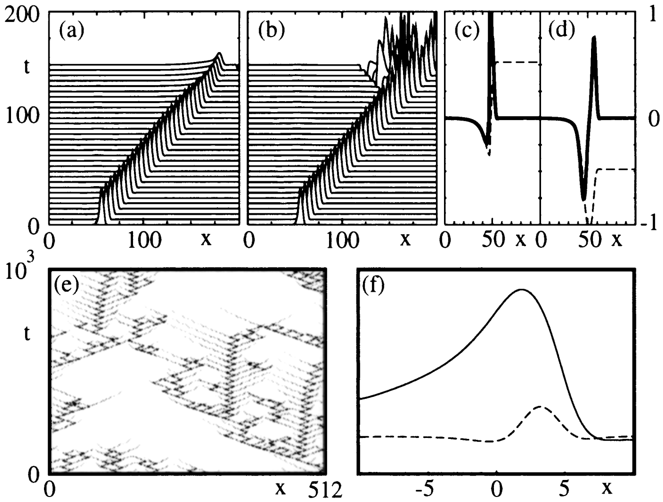

FIG. 4. (a),(b) Evolution of the wave-number profiles of perturbed (perturbation $\sim 10^{-6}$) homoclinic holes. Consecutive time slices have a time difference of 5, (c),(d) Wave-number profiles (solid curves) and winding number $\int dxq/2\pi$ (dashed curves) just before (c) and after (d) the first phase slip. (e) The zigzaging holes near the transition to plane waves ($c_1 = 0.6, c_3 = 1.28$). (f) The real (solid curve) and imaginary (dashed curve) part of the linearly unstable mode.

The homoclinic holes are, for our choice of parameters, clearly unstable. Similar to the local structures found in the intermittent state, they either slowly decay or grow out to a phase slip [Figs. 4(a) and 4(b)]. When there are no phase slips, the total phase difference $\Delta\theta \approx 3.24$ across the hole is conserved. The decaying hole is shown in Fig. 4(a); the wave number peak, amplitude dip, and apparent velocity decrease; and for long times, the dynamics crosses over to a slow phase diffusion by which $\Delta\theta$ is smeared out. In Fig. 4(b), the evolution of a homoclinic hole towards phase slips is shown. The wave number peak and amplitude dip slowly grow, and at $t \approx 117$ the first phase slip occurs, which nucleates a typical intermittent state. Before this phase slip, the wave number acquires a negative peak in order to conserve the total phase difference across the structure [Fig. 4(c)]. Both of these peaks diverge at the phase-slip event and, just after the phase slip, the winding number $\int dxq/2\pi$ decreased by 1. Therefore, the negative phase bump that corresponds to the new left moving hole is quite steep [Fig. 4(d)], and this hole will quickly grow out to a new phase slip, from which a strong right moving hole is generated, etc. When we quench c_1 and c_3 in the direction of the transition to plane waves, these zigzag motions of the holes become very dominant [Fig. 4(e)].

Since the asymptotic $q = 0$ waves are linearly stable, the unstable modes can be expected to be localized "core" modes. Following Aranson [22] we study an ansatz of the form $A(\xi) = e^{-i\omega t}[a(\xi) + e^{\lambda t}w(\xi)]e^{i\phi(\xi)}$, where w is an infinitesimal, complex-valued perturbation. This yields (assuming that λ is real) a set of seven coupled ODE's; three for a, q, and κ describing the homoclinic hole, and four for the real and imaginary parts of w and $\partial_\xi w$. Solving these ODE's by a shooting algorithm, we ascertain that there is *one* unstable core mode with eigenvalue $\lambda = 0.0929$ [Fig. 4(f)]. The spatial decay rate of the trailing edge of this mode results from the essentially unaffected propagation of the holes; indeed we find here $w \sim e^{\lambda x}$. To check w and λ we have verified that direct simulations of the CGLE, as shown in Figs. 4(a) and 4(b), yield similar w and λ as the ones obtained by the shooting. As an extra check, observe that phase conservation of the CGLE yields that $W := \int dx \arg(w)$ should be zero; the numerics yield $W < 10^{-4}$.

The fact that the homoclinic holes have only *one* weakly unstable mode is reflected in the dynamics in the intermittent regime. The key point is that most sufficiently localized wave number blobs will be attracted to the 1D unstable manifold; subsequently, they then evolve along this manifold, in either the "decay" or the "phase slip" direction. We can loosely think of the homoclinic holes as unstable equilibria between plane waves and phase slips. To illustrate this consider Fig. 5(a), where we follow an initial condition with $|A| = 1$ and a triangular wave number profile. We are able to let this rather arbitrary initial condition evolve to a homoclinic hole by adjusting only *one* parameter in the initial condition (the height of the triangle was set to 0.437 754 while its width was 20). Increasing the height or width of the initial wave number blob leads to a steepening hole [Fig. 4(b)], while a decrease leads to a decaying hole [Fig. 4(a)].

The two wave number blobs formed after a phase slip [Fig. 4(d)] are also attracted to the unstable manifold. As a result, the local structures in the spatiotemporal intermittent regime evolve essentially *along the 1D unstable manifold* of the $q = 0$ homoclinic holes. This claim is substantiated by the fact that the values of the extrema of q and the corresponding local minima of $|A|$, obtained from a long run in the intermittent regime, are strongly correlated [Fig. 5(b)]. This indicates that a one-parameter

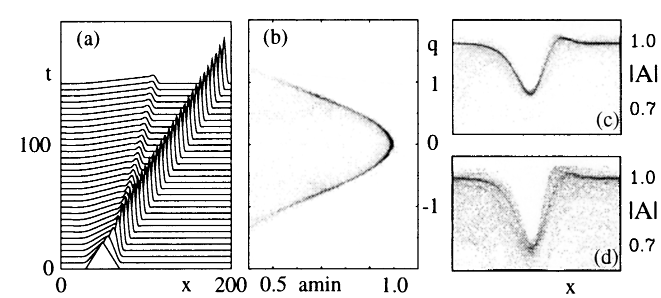

FIG. 5. (a) The wave-number profiles of a "triangular" initial condition evolving to a homoclinic hole. The total phase difference across the triangular initial condition is 8.76, so a big positive wave-number packet is emitted. (b) Collapse of the extrema of a and q in the intermittent regime. (c) Amplitude profiles of $|A|$ with minima of $|A|$ at about the homoclinic hole value. (d) Similar profiles for steeper minima.

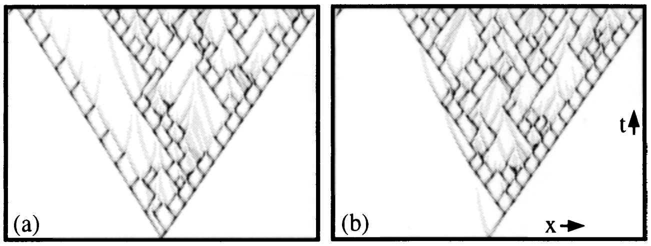

FIG. 6. (a) Space-time (512×250) plot of $|A|$, showing homoclinic holes in a background state with wave number 0.021. Because of the high contrast, both holes (black curves) and small wave packets (grey) are visible. Note the "bound state" to the left. (b) Similar for wave number $= 0.035$

family of profiles of A is dominant. In Figs. 5(c) and 5(d) we collapsed the profiles of $|A|$ corresponding to certain minima. In Fig. 5(c) we require the minimum to be about 0.78, and the profiles then correspond to the coherent homoclinic hole, while in Fig. 5(d) we focus on steeper minima and obtain profiles of states evolving towards phase slips.

All of this suggests a phenomenological model in terms of moving "homoclinic hole" particles, where each particle possesses an internal degree of freedom that parametrizes its location on the unstable manifold. However, the sensitivity of the holes to the wave number of the laminar patches they invade complicates particle models (Fig. 6). Note that the homoclinic holes are neither sources nor sinks because their propagation velocity is much larger than the typical group velocity of the surrounding waves. Suppose we follow a hole that moves to the right into a state with small wave number q_r. When $q_r > 0$, this leads to the "winding up" of the hole (increase of $\Delta\theta$), which pushes the hole towards phase slips (Fig. 6). When $q_r < 0$ or, equivalently, when a left moving hole invades a state with positive q, this leads to the "winding down" of the hole, which delays the phase slips [Fig. 6(a)] or pushes the hole towards decay [Fig. 6(b)] [23]. This strong sensitivity to the asymptotic waves has as a consequence that in the intermittent regime, small wave packets, resulting from, for instance, decaying holes, strongly affect the lifetime of the propagating holes. This coupling of the background to the holes seems to be an important mechanism for generating chaos; when the holes invade homogeneous states (as is the case for the edge holes in Fig. 6), the dynamics appears rather regular.

In conclusion, we have described a new class of coherent solutions which occur in several regimes of the 1D CGLE and which are intimately connected to phase slips.

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