Mean Dynamical Entropy of Quantum Maps on the Sphere Diverges in the Semiclassical Limit

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We analyze quantum dynamical entropy based on the notion of coherent states. The mean value of this quantity for quantum maps on the sphere is computed as an average over the uniform measure on the space of unitary matrices of size N. Mean dynamical entropy is positive for $N \geq 3$, which supplies a direct link between random matrices of the circular unitary ensemble and the chaotic dynamics of the corresponding classical maps. Mean entropy tends logarithmically to infinity in the semiclassical limit $N \to \infty$ and this indicates the ubiquity of chaos in classical mechanics. [S0031-9007(98)05487-8]

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Quantum analogs of classically chaotic systems have been an object of intensive investigations for almost twenty years. The study of statistical properties of the spectra of quantized chaotic systems is for the purpose of trying to prove that these systems can be described by suitable ensembles of random matrices [1–3]. In this paper we follow the opposite direction: Studying a generic quantum system we find support of the conclusion that the dynamical entropy of the corresponding classical system is positive and, actually, arbitrary large. More precisely, we analyze the set of all *structureless* quantum systems [4] (without geometric or time reversal symmetries). For these systems, described by the circular ensemble of unitary matrices, we compute the mean dynamical entropy averaged over the Haar measure and show that it increases logarithmically with the dimension of the Hilbert space. We discuss quantum analogs of the classical area preserving maps on the sphere. To link the quantum dynamics with the classical phase space, one uses in this case the well-known SU(2) spin coherent states.

A classical dynamical system is called *chaotic*, if its Kolmogorov-Sinai (KS) entropy is positive. However, this definition cannot be applied literally to quantum systems, since a widely accepted generalization of KS entropy for quantum mechanics has not yet been found. Several attempts to define such a quantity are known [5–7]. However, some of them, such as the Connes-Narnhofer-Thirring entropy [8] or the Alicki-Fannes [9] entropy, vanish for finite-dimensional quantum systems, and can be applied rather in quantum statistical mechanics. Others do not give the correct semiclassical limit.

In a series of papers [6,10,11] we proposed a new definition of dynamical quantum entropy based on the notion of coherent states. Our approach relies on the assumption that the knowledge of the time evolution of a quantum state is obtained by performing a sequence of approximate quantum measurements. The evolution of the system between two subsequent measurements is governed by a unitary matrix U, but the sequence of measurements introduces a nonunitary evolution of the system [10].

Let us consider a classical area preserving map on the sphere $\Theta : S^2 \to S^2$ and a corresponding quantum map U acting in an N-dimensional Hilbert space $\mathcal{H}$. A link between classical and quantum mechanics can be established via a family of spin coherent states $|x\rangle \in \mathcal{H}$ localized at points x of the sphere. These SU(2) coherent states can be defined as [12,13] $|x\rangle = R_x|\kappa\rangle$, where R_x are the rotation operators and the reference state $|\kappa\rangle$ is usually taken as the maximal eigenstate $|j, j\rangle$ of the component J_z of the angular momentum operator. The identity resolution reads $\int_{S^2} |x\rangle\langle x|\, dx = I$, where dx is the uniform measure on the sphere. For our purposes it is convenient to normalize coherent states as $\langle x \mid x\rangle \equiv N = 2j + 1$.

To work with the coherent states entropy we need to consider a partition (coarse graining) $\mathcal{A} = \{E_1, \dots, E_k\}$ of the phase space, where the sum of volumes of all cells is normalized to unity $[\sum_{i=1}^{k} \mathrm{vol}(E_i) = 1]$. The partition generates the symbolic dynamics in the k-symbol code space. The results of sequential approximate quantum measurement are represented by the strings of n letters $\iota = \{i_0, \dots, i_{n-1}\}$, where each letter i_j denotes one of the k cells. The probabilities P_ι^{CS} of entering the cells $E_{i_0}, ..., E_{i_{n-1}}$ can be expressed by the integrals

$$P_\iota^{\mathrm{CS}} := \int_{E_{i_0}} dx_0 \cdots \int_{E_{i_{n-1}}} dx_{n-1} \times \prod_{u=1}^{n-1} K_U(x_{u-1}, x_u)\,, \tag{1}$$

while the kernel K_U is given in terms of coherent states

$$K_U(x,y) := \frac{1}{N}\,|\langle y|U|x\rangle|^2 \tag{2}$$

for $x, y \in S^2$ [6]. The kernel $K_U(x,y)$ may be thus interpreted as a y-dependent *Husimi distribution* (Q-function) of the transformed state $U|x\rangle$. If U equals the identity operator I, the quantity $K_I(x,y)$ is called the *overlap* of coherent states $|x\rangle$ and $|y\rangle$.

In close analogy with the classical KS entropy we define the *coherent states* (CS) *entropy of U with respect*

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to the partition $\mathcal{A}$

$$H(U,\mathcal{A}) := \lim_{n\to\infty}(H_{n+1} - H_n) = \lim_{n\to\infty}\frac{1}{n}H_n\,, \qquad (3)$$

where the partial entropies H_n are given by the sum over all k^n strings ι of length n

$$H_n := \sum_\iota -P_\iota^{\rm CS}\ln P_\iota^{\rm CS}. \qquad (4)$$

Note that both sequences in (3) are decreasing and the quantity $H_1 = -\sum_{i=1}^k \mathrm{vol}(E_i)\ln[\mathrm{vol}(E_i)]$, which does not depend on U, is just the *entropy of the partition* $\mathcal{A}$. We denote it by $H(\mathcal{A})$.

There are two kinds of randomness in our model: The first is connected with the underlying unitary dynamics of the system; the second comes from the approximate measurement process. Accordingly, we split the partition dependent CS entropy into two components: *CS measurement entropy* and *CS dynamical entropy*:

$$H_{\rm meas}(\mathcal{A}) : = H(I,\mathcal{A})\,, \qquad (5)$$

$$H_{\rm dyn}(U,\mathcal{A}) : = H(U,\mathcal{A}) - H_{\rm meas}(\mathcal{A}). \qquad (6)$$

In order to keep away from ambiguity in the choice of the partition we define *CS dynamical entropy of U* as

$$H_{\rm dyn}(U) := \sup_{\mathcal{A}} H_{\rm dyn}(U,\mathcal{A})\,, \qquad (7)$$

the supremum being taken over all finite partitions.

In [10,11] we study the properties of CS dynamical entropy and present the methods of its numerical computing based on the concept of iterated function systems. It is conjectured that in the semiclassical limit $N\to\infty$ the CS dynamical entropy of a family of quantum maps U_N tends to the KS entropy of the corresponding classical map θ, if certain assumptions linking classical and quantum maps are fulfilled [6]. Recent numerical calculation shows [14] that for some quantum analogs of classically chaotic maps on the sphere (kicked top, baker map on the sphere) the CS dynamical entropy is positive, grows monotonically with the dimension of the Hilbert space N, and is smaller than the KS entropy of the corresponding classical map. Since a scheme of discrete approximate measurements leads to a nonunitary time evolution of the system [10], the CS dynamical entropy of such quantum maps remains positive in the time limit (3), in contrast with the entropy introduced in [15].

In this Letter we evaluate the mean value of CS dynamical entropy $\langle H_{\rm dyn}(U)\rangle_{U(N)}$, taking the average over the unitary matrices $U(N)$ of the circular unitary ensemble (CUE). Computing the CS dynamical entropy requires the time limit $n\to\infty$. Surprisingly, one can obtain bounds for this quantity by analyzing the *continuous entropy* of U, which depends only on the one-step evolution of the quantum system:

$$H_U := -\int_{S^2}\int_{S^2} K_U(x,y)\ln K_U(x,y)\,dx\,dy\,. \qquad (8)$$

This quantity is related to the "classical-like" entropy introduced into quantum mechanics by Wherl [16]. Namely, H_U is equal to the difference of the Wherl entropy of the states $U|x\rangle$, averaged over all points x on the sphere, and $\ln N$ [the latter term follows from the normalization in (2)]. Similar quantities have also been studied by Schroeck [17] and by Mirbach and Korsch [18]. Calculation of the continuous entropy is particularly easy for the identity operator $U = I$ and gives the Wehrl entropy of a single coherent state [19]

$$H_I = -\ln N + \frac{N-1}{N}\,. \qquad (9)$$

We shall proceed toward an estimate of the partition-dependent CS entropy (3) for an arbitrary unitary matrix U. Using classical methods from the information theory (see [20], Sect. 2.2) we obtain

$$\inf_{\mathcal{A}}[H_{n+1}(U,\mathcal{A}) - H_n(U,\mathcal{A}) - H(\mathcal{A})] = H_U \qquad (10)$$

for each natural n, where the coherent states partial entropies $H_n(U,\mathcal{A})$ are defined by (4). From the definition of CS dynamical entropy we get

$$H_U + H(\mathcal{A}) \le H(U,\mathcal{A}) \le H(\mathcal{A}) \qquad (11)$$

and

$$\inf_{\mathcal{A}}[H(U,\mathcal{A}) - H(\mathcal{A})] = H_U\,. \qquad (12)$$

In fact, the infimum in (10) and (12) is achieved if the maximal diameter of a member of the partition $\mathcal{A}$ tends to zero. Thus, for a sufficiently fine partition, the dynamical CS entropy splits into approximately two parts: $H(\mathcal{A})$ which depends only on the partition, and H_U depending only on the dynamics. Combining the above formulas with the analogous one obtained for $U = I$, we conclude that

$$-H_I + H_U \le H_{\rm dyn}(U) \le -H_I\,. \qquad (13)$$

The famous Lieb conjecture [19] states that the Werhl entropy attains its minimum (9) for any coherent state (for partial results, see [21]). This would imply $H_I \le H_U$, and consequently $H_{\rm dyn}(U) \ge 0$ for every unitary matrix U. As we can see above, the quantity H_I decreases approximately as $-\ln N$ and so, if the Lieb conjecture is true, then the entropy $H(U,\mathcal{A})$ is limited from below by $H(\mathcal{A}) - \ln N$. This agrees with the bound obtained by Halliwell [22] for the information of the phase space distributions derived from the probabilities for quantum histories. Note, however, that the bound (11) seems to be much more precise, because, as we will show, $-H_U$ is typically much smaller then $-H_I$.

In order to estimate the mean entropy of quantum maps we average (13) over the space of unitary matrices $U(N)$ with respect to the Haar measure μ,

$$-H_I + \langle H_U\rangle_{{\rm U}(N)} \le \langle H_{\rm dyn}(U)\rangle_{{\rm U}(N)} \le -H_I\,. \qquad (14)$$

Thus, to obtain the desired bounds for the mean CS dynamical entropy, it suffices to calculate $\langle H_U\rangle_{{\rm U}(N)}$.

We have

$$\langle H_U \rangle_{\mathrm{U}(N)} = -\int_{\mathrm{U}(N)} \left(\int_{S^2} \int_{S^2} K_U(x,y) \ln K_U(x,y)\, dx\, dy \right) \times d\mu(U). \tag{15}$$

Since $K_U(x,y) = |\langle y|U|x\rangle|^2/N = |\langle \kappa | T_y^{-1} U T_x | \kappa \rangle|^2/N$, one may interchange the order of integration and use the invariance of the Haar measure. Putting $V := T_y^{-1} \mathrm{U} T_x$ we conclude that

$$\langle H_U \rangle_{\mathrm{U}(N)} = -\int_{\mathrm{U}(N)} \frac{|\langle \kappa|V|\kappa\rangle|^2}{N} \ln \frac{|\langle \kappa|V|\kappa\rangle|^2}{N}\, d\mu(V). \tag{16}$$

To calculate this quantity we make use of the formulas for the distribution of $\langle \kappa|U|\kappa\rangle$ given by Kuś *et al.* [23]. Otherwise, we can refer to the results of Jones [24]. Applying one of these methods, we get

$$|H_U\rangle_{\mathrm{U}(N)} = -\ln N + \Psi(N+1) - \Psi(2), \tag{17}$$

where Ψ denotes the digamma function, which for natural arguments $k < n$ satisfies $\Psi(n) - \Psi(k) = \sum_{l=k}^{n-1} \frac{1}{l}$.

Finally from (9), (14) and (17) we obtain the main result of this work: *a lower and an upper bound for the mean CS dynamical entropy*

$$\Psi(N+1) - \Psi(2) - 1 + \frac{1}{N} \le \langle H_{\mathrm{dyn}} \rangle_{\mathrm{U}(N)}, \qquad \langle H_{\mathrm{dyn}} \rangle_{\mathrm{U}(N)} \le \ln N - 1 + \frac{1}{N}. \tag{18}$$

The difference between an upper bound (which is the maximal value of the CS dynamical entropy) and a lower one converges to the constant $1 - \gamma \simeq 0.42278$ if $N \to \infty$. Hence the mean value of CS dynamical entropy tends in the semiclassical limit to the infinity exactly as $\ln N$. The dependence of both bounds on the quantum number $N = 2j + 1$ is presented in Fig. 1. In the semiclassical limit $N \to \infty$ the mean dynamical entropy diverges in contrast to the CS dynamical entropy of a given quantum map, which seems to converge to the KS entropy of the corresponding classical system. Therefore, for sufficiently large N, a matrix F_N representing a given quantum map will not be generic with respect to the Haar measure on $\mathrm{U}(N)$.

To visualize this difference we present in Fig. 2 the Husimi function of an exemplary coherent state $|\vartheta, \phi\rangle = |0.93, 3.30\rangle$ transformed once by a Floquet operator $F = e^{ipJ_z} e^{iKJ_x^2/2j}$ representing the kicked top [2] with $j = 15\frac{1}{2}$ in a classically chaotic regime ($p = 1.7, K = 7$) (a), and by a random unitary matrix U (b) [25]. The sphere is represented in the Mercator projection with $0 \le \phi < 2\pi$ and $0 \le \vartheta < \pi$, $t = \cos\vartheta$. In the former case, the wave packet remains localized in the vicinity of the classical trajectory, while in the latter, it is already entirely delocalized after one iteration. The same data plotted in the log scale allow one to detect the zeros of Husimi functions [26]. For the quantum map F they form a regular spiral-like structure (c), in contrast to the random distribution over the entire phase space for the random matrix U (d).

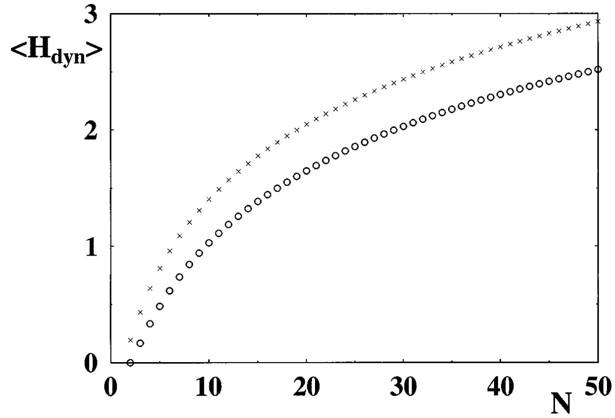

FIG. 1. Upper ($\times$) and lower ($\circ$) bounds for the mean CS dynamical entropy of unitary matrices representing structureless quantum systems on the sphere S^2 as a function of the matrix dimension $N = 2j + 1$.

Consider a more general operator $\hat{F} = e^{ipH} e^{iKH'}$, where H and H' are noncommuting Hermitian operators constructed as polynomials of a given order M in J_x, J_y, J_y. For generic values of p and K, one may thus expect that the N-dimensional representations of $\hat{F}$ are characterized by the CS dynamical entropy typical to random matrices only for $N \le M$. However, in the semiclassical limit, one increases N keeping M constant.

The above results, obtained for the sphere S^2 and the SU(2) spin coherent states, can be generalized for classical

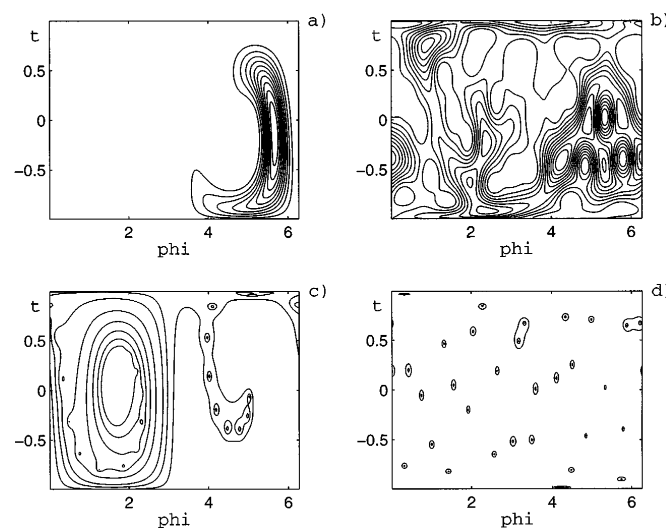

FIG. 2. Contour plot of the Husimi function of an exemplary coherent state transformed by the quantum kicked top map (a) and by a generic random matrix (b) of size $N = 32$. The zeros of Husimi function are visible in (c) and (d), respectively, obtained from the same data using a log scale for the contour heights.

phase spaces associated with higher groups SU(d); $d \geq 2$, which are the complex projective spaces CP^{d-1} in this case. The dimension of the Hilbert space is then $N = \dim(\mathcal{H}) = \binom{m+d-1}{m}$ with $m = 1, 2, \ldots$, while $|x\rangle$ represents the SU(d) coherent states [13]. The Wehrl entropy of such a coherent state equals

$$H_I = -\ln N + m[\Psi(m + d) - \Psi(m + 1)]. \quad (19)$$

Following Lieb [19] we conjecture that this value gives theminimal Wehrl entropy for SU(d). Performing the steps similar to (10)–(17), we arrive at bounds for the mean CS dynamical entropy analogous to (18)

$$l_b \leq \langle H_{\mathrm{dyn}}^{\mathrm{SU}(d)} \rangle_{\mathrm{U}(N)} \leq u_b\,, \quad (20)$$

$$\begin{aligned} l_b &= \Psi(N + 1) - \Psi(2) \\ &\quad - m[\Psi(m + d) - \Psi(m + 1)], \quad (21) \\ u_b &= \ln N - m[\Psi(m + d) - \Psi(m + 1)]. \end{aligned}$$

In the semiclassical limit $m \to \infty$ we get a simple approximation for both bounds: $l_b \sim \ln N - d + \gamma$ and $u_b \sim \ln N - d + 1$, where γ is the Euler constant.

Obtained estimates (18) and (20), and (21) allow us to conclude that a quantum system represented by a typical unitary matrix from CUE is characterized by positive dynamical entropy, which is only insignificantly smaller than the maximal one diverging with $m \sim 1/\hbar$. In other words, a generic quantum system is almost as chaotic, as possible. We proved this for SU(d) coherent states, but the method seems to work also in the general case, i.e., for coherent states defined on arbitrary homogeneous compact manifold, as well as for the orthogonal and symplectic circular ensembles.

At first glance, this result seems to be paradoxical as the KS entropy of a classical map is finite and the CS dynamical entropy of the corresponding quantum system seems to tend to this value in the semiclassical limit. Hence for a Hilbert space of a sufficiently large dimension, matrices representing a quantum analog of a given classical chaotic system cannot be typical. Their entropy is substantially smaller than the CUE average, even though many other statistics (two point correlations, level spacing distribution, spectral rigidity [1,2]) conform to the predictions of random matrix theory.

However, this need not contradict the general belief that quantum analogs of classically chaotic systems might be represented by typical unitary matrices. The paradox could be resolved if we assume that strongly chaotic systems dominate less chaotic ones in the “space” of classical systems defined on the corresponding symplectic manifold. Thus, our results provide a strong argument in favor of the ubiquity of chaos in classical mechanics.

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