Generalization of a Fermi Liquid to a Liquid with Fractional Exclusion Statistics in Arbitrary Dimensions: Theory of a Haldane Liquid

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A liquid of interacting quasiparticles with Haldane-Wu (fractional exclusion) statistics in arbitrary dimensions is discussed in terms of Fermi-liquid theory. The universal properties of this Haldane liquid are investigated. [S0031-9007(97)05241-1]

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In 1957 Landau [1] introduced the famous concept of quasiparticles with Fermi-Dirac statistics (FDS) in a Fermi liquid in order to describe electrons in a metal where the interaction between electrons was believed to be very significant, as a generalization of the Sommerfeld theory for a degenerate Fermi gas [2]. This description was later justified by Luttinger [3].

Recently Haldane [4] introduced the concept of fractional exclusion statistics (FES) which quasiparticles in strongly interacting systems in arbitrary dimensions might satisfy as a generalization of Pauli's exclusion principle. Wu [5] first formulated quantum statistical mechanics (QSM) in the state representation and derived the distribution function for an ideal gas with FES as a generalization of the FD and the Bose-Einstein (BE) distribution functions. [Throughout this Letter the FES is referred to as the Haldane-Wu statistics (HWS) such as FDS and BES, while a gas or liquid with HWS is called a Haldane gas or liquid such as a gas or liquid with FDS (BES) was called a Fermi (Bose) gas or liquid.]

This concept has played a very important role to understand strongly interacting system such as the Tomonaga-Luttinger model (TLM) [6], the Calogero-Sutherland model (CSM) [7], and the Haldane-Shastry model (HSM) [8] in one dimension and the fractional quantum Hall effect in two dimensions [9]. Especially, notable is that Dasnières de Veigy and Ouvry [10] revealed a deep connection between the FES and the fractional quantum Hall system, while Bernard and Wu [11] and Isakov [12] found the one between the FES and the CSM. More recently, QSM formulation has been developed further by Nayak and Wiczek [13], Isakov, Arovas, Myrheim, and Polychronakos [14], and the author [15]. Here, the QSM formulation allows us to evaluate the equation of state for an ideal gas with HWS in arbitrary dimensions with obtaining all the exact cluster coefficients in the cluster expansion [14,15].

However, interacting quasiparticles with HWS in arbitrary dimensions have never been investigated yet, and therefore the concept of a Haldane liquid is still missing. In this Letter I will explore this concept as a generalization of the Landau's Fermi liquid theory and generalize the Luttinger's theorem for a Fermi liquid to that for a Haldane liquid.

To ease the reference I first summarize the universal properties of a Haldane liquid stemmed from the point of view of quasiparticles in the Landau's Fermi liquid theory [1] as follows: (1) the particle-hole asymmetry universally exists in the system; (2) the volume of the pseudo-Fermi sphere is conserved under the introduction of the interaction between quasiparticles—the generalized Luttinger's theorem from a Fermi liquid [3] to a Haldane liquid; (3) many physical quantities such as the specific heat are T linear at very low temperature unless $g = 0$ [13–15]; and (4) the true condensation exists only when $g = 0$ [15]. Here g stands for a pure HWS parameter [see Eq. (1)].

In this way many properties of the Haldane liquid are shared with those of the Luttinger liquid in one dimension such as the TLM [6], CSM [7], and HSM [8] as a generalization of the Fermi liquid [1]. Thus, I conclude that the concept of a Haldane liquid presented here is a generalization of that of a Luttinger liquid as well as a Fermi liquid.

Let us first consider the system of a Haldane gas with a statistical parameter g. The total number N, the energy E and the entropy S of the system of quasiparticles are given by $N = \sum_{\mathbf{p}} n_{\mathbf{p}}$, $E = \sum_{\mathbf{p}} \epsilon_{\mathbf{p}} n_{\mathbf{p}}$, $S = k_B \sum_{\mathbf{p}} (n_{\mathbf{p}} + \rho_{\mathbf{p}}) \ln(n_{\mathbf{p}} + \rho_{\mathbf{p}}) - n_{\mathbf{p}} \ln n_{\mathbf{p}} - \rho_{\mathbf{p}} \ln \rho_{\mathbf{p}}$, respectively, where $n_{\mathbf{p}}$ is the momentum distribution function of quasiparticles and $\rho_{\mathbf{p}}$ the hole distribution function with k_B the Boltzmann constant. If I impose the condition of HWS with g,

$$n_{\mathbf{p}} + \rho_{\mathbf{p}} = 1 - (g - 1)n_{\mathbf{p}}\,, \qquad (1)$$

then by taking the extreme of the thermodynamic potential Ω as $\delta\Omega = \delta(E - \mu N - TS) = 0$, I obtain $\Omega = -PV = -k_B T \sum_{\mathbf{p}} \ln[(1 + W_{\mathbf{p}})/W_{\mathbf{p}}]$, $N = \sum_{\mathbf{p}} n_{\mathbf{p}} = \sum_{\mathbf{p}} 1/(W_{\mathbf{p}} + g)$, where I have the Wu's functional relation [4],

$$W_{\mathbf{p}}^{g}(1 + W_{\mathbf{p}})^{1-g} = e^{\beta(\epsilon_{\mathbf{p}} - \mu)}. \qquad (2)$$

This is an alternative derivation for the thermodynamic potential and the density of the system compared to the derivation in Ref. [15], and has been discussed by Bernard and Wu [11] as a conjecture generalizing the CSM to the models in higher dimensions. I also note that if

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one identifies as $\rho_{\mathbf{p}}/n_{\mathbf{p}} = W_{\mathbf{p}}$ and substitutes this into Eq. (1), one obtains $n_{\mathbf{p}} = 1/(W_{\mathbf{p}} + g)$.

Let us now consider the interaction between quasiparticles. Following Landau [1], not only $\epsilon_{\mathbf{p}}$ for a given distribution $n_{\mathbf{p}}$ but also the change in $\epsilon_{\mathbf{p}}$ produced by a change in $n_{\mathbf{p}}$ is of essential importance for the theory of the quantum liquid. Hence, I assume that the energy change is given as

$$\delta\epsilon_{\mathbf{p}} = \sum_{\mathbf{p}'} f(\mathbf{p},\mathbf{p}')\delta n_{\mathbf{p}'}, \tag{3}$$

where the function $f(\mathbf{p},\mathbf{p}')$ is a symmetric function relative to $\mathbf{p}$ and $\mathbf{p}'$ and called the scattering matrix describing the scattering between quasiparticles such as the scattering process: $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_1' + \mathbf{p}_2'$. I assume that *the velocity of the quasiparticle in a quantum liquid is given by* $\partial\epsilon_{\mathbf{p}}/\partial\mathbf{p}$, *and the number of quasiparticles coincides with the number of free quasiparticles in a quantum gas*, from which I have the total momentum given by

$$\mathbf{P} = \sum_{\mathbf{p}} \mathbf{p}_0 n_{\mathbf{p}} = \sum_{\mathbf{p}} m \frac{\partial\epsilon_{\mathbf{p}}}{\partial\mathbf{p}} n_{\mathbf{p}}, \tag{4}$$

where $\mathbf{p}_0$ represents momentum for a free quasiparticle when no interaction exists and it is a function of $\mathbf{p}$. The variational derivation with respect to $n_{\mathbf{p}}$ should be the same on both sides of Eq. (4). Then using Eq. (3) I obtain

$$\frac{1}{m}\sum_{\mathbf{p}} \mathbf{p}_0 \delta n_{\mathbf{p}} = \sum_{\mathbf{p}} \frac{\partial\epsilon_{\mathbf{p}}}{\partial\mathbf{p}}\delta n_{\mathbf{p}} + \sum_{\mathbf{p}}\sum_{\mathbf{p}'}\frac{\partial}{\partial\mathbf{p}} f(\mathbf{p},\mathbf{p}')\delta n_{\mathbf{p}'} n_{\mathbf{p}}.$$

Since the $\delta n_{\mathbf{p}}$ is arbitrary, I obtain

$$\frac{\mathbf{p}_0}{m} = \frac{\partial\epsilon_{\mathbf{p}}}{\partial\mathbf{p}} + \sum_{\mathbf{p}'}\frac{\partial f(\mathbf{p}',\mathbf{p})}{\partial\mathbf{p}'} n_{\mathbf{p}'}. \tag{5}$$

Denoting $m\partial\epsilon_{\mathbf{p}}/\partial\mathbf{p}$ by $\mathbf{p}$, the Landau's relation Eq. (5) turns out to be

$$\mathbf{p} = \mathbf{p}_0 + \mathbf{f}(\mathbf{p}), \tag{6}$$

$$\mathbf{f}(\mathbf{p}) \equiv -\sum_{\mathbf{p}'} \mathbf{f}(\mathbf{p}',\mathbf{p}) n_{\mathbf{p}'}, \qquad \mathbf{f}(\mathbf{p}',\mathbf{p}) \equiv m\frac{\partial f(\mathbf{p}',\mathbf{p})}{\partial\mathbf{p}'}. \tag{7}$$

This type of relations was recently discussed by Sutherland [16] as a generalization of the CSM in one dimension to that in two dimensions, and the $\mathbf{f}(\mathbf{p})$ was called the *displacement field.* Here the scattering function $f(k) = f(|k|)$ with $\hbar = 1$ was represented in terms of the two-dimensional partial wave phase shifts $\theta_l(k)$ such that $kf(k) = 4\sum_{l=-\infty}^{\infty}\theta_{2l+1}(k)$ [see Eq. (3.7) in Ref. [16]]. And he defined a generalization of Yang and Yang's method [17] to that in the higher-dimensional systems by taking the derivative of Eq. (6) with respect to $\mathbf{p}$, which then gives the relation

$$n_{\mathbf{p}} + \rho_{\mathbf{p}} \equiv \frac{d^D p_0}{d^D p} = \left|\frac{\partial\mathbf{p}_0}{\partial\mathbf{p}}\right| = \det\left[\mathbf{1} - \frac{\partial\mathbf{f}(\mathbf{p})}{\partial\mathbf{p}}\right], \tag{8}$$

where $|\partial\mathbf{p}_0/\partial\mathbf{p}|$ means the Jacobian. If I expand Eq. (8) up to linear order of $\mathbf{f}(\mathbf{p})$, then I obtain

$$n_{\mathbf{p}} + \rho_{\mathbf{p}} = 1 - \sum_{\mathbf{p}'}[g(\mathbf{p}',\mathbf{p}) - \delta(\mathbf{p} - \mathbf{p}')]n_{\mathbf{p}'}, \tag{9}$$

$$g(\mathbf{p}',\mathbf{p}) \equiv \delta(\mathbf{p} - \mathbf{p}') - \frac{\partial\mathbf{f}(\mathbf{p}',\mathbf{p})}{\partial\mathbf{p}}. \tag{10}$$

Here $\delta(\mathbf{p} - \mathbf{p}')$ is a Dirac's delta function and the contribution of $\partial\mathbf{f}(\mathbf{p},\mathbf{p}')\partial\mathbf{p}$ from $\mathbf{p} = \mathbf{p}'$ is excluded by the delta-function part. This can be regarded as the generalization of the pure HWS of Eq. (1) to the mutual HWS case. Thus, *Eq. (6) determines the statistics of the system.*

As Landau [1] also discussed the quasiparticle energy changes due to the change of the momentum distribution function, from Eq. (3) the quasiparticle energy $\epsilon_{\mathbf{p}}$ is given by

$$\epsilon_{\mathbf{p}} = \epsilon_{\mathbf{p}}^{(0)} + \sum_{\mathbf{p}'} f(\mathbf{p},\mathbf{p}')n_{\mathbf{p}'}. \tag{11}$$

This gives the famous free energy functional of the mean-field type as

$$F - \mu N = \sum_{\mathbf{p}}(\epsilon_{\mathbf{p}}^{(0)} - \mu)n_{\mathbf{p}} + \frac{1}{2}\sum_{\mathbf{p}}\sum_{\mathbf{p}'} f(\mathbf{p},\mathbf{p}')n_{\mathbf{p}}n_{\mathbf{p}'} - TS. \tag{12}$$

When Landau [1] considered quasiparticles in a Fermi liquid, there existed only the concept of FD or BE quasiparticles. Therefore, the momentum distribution function might have been either $n_{\mathbf{p}} = 1/(e^{\beta(\epsilon_{\mathbf{p}}-\mu)} + 1)$ for FDS or $n_{\mathbf{p}} = 1/(e^{\beta(\epsilon_{\mathbf{p}}-\mu)} - 1)$ for BES. However, this is not true in our case of quasiparticles with HWS in a Haldane liquid. In this case, Eq. (12) together with Eq. (9) must be maximized, which provides the desired Wu's distribution function Eq. (2).

The free energy functional of Eq. (12) was also found recently by Haldane [18] in the study of the HSM [8] and by Murthy and Shankar [19] in the study of the CSM [7], respectively. In the former the excitations are described as the spinon gas represented by an exact mean-field theory with BES [see Eq. (8) in Ref. [18]], while in the latter the excitations are described as quasiparticles represented by a mean-field theory with HWS [see Eq. (5) in Ref. [19]]. Thus, I conjecture that *there exists a class of interacting quasiparticle systems with HWS in any dimension such that the above Eq. (12) becomes exact.*

Let us consider the relationship between Eq. (11) and Yang and Yang's method [17]. In the Landau's quasiparticle picture, if I assume that quasiparticles are fermions with the FD momentum distribution function, then I have $\epsilon_{\mathbf{p}} = \epsilon_{\mathbf{p}}^{(0)} + \sum_{\mathbf{p}'} f(\mathbf{p}',\mathbf{p})/(e^{\beta(\epsilon_{\mathbf{p}'}-\mu)} + 1)$, which is an integral equation for $\epsilon_{\mathbf{p}}$. Assuming as

$$\phi(\mathbf{p}',\mathbf{p}) \equiv \frac{\partial f(\mathbf{p}',\mathbf{p})}{\partial\epsilon_{\mathbf{p}'}}, \tag{13}$$

the integration by parts yields

$$\epsilon_{\mathbf{p}} = \epsilon_{\mathbf{p}}^{(0)} + k_B T\sum_{\mathbf{p}'}\phi(\mathbf{p},\mathbf{p}')\ln(1 + e^{-\beta(\epsilon_{\mathbf{p}'}-\mu)}). \tag{14}$$

This is exactly the same form known as the *thermal Bethe ansatz (TBA) equation* in Yang and Yang's method [17] for the one-dimensional systems where $\phi(\mathbf{p},\mathbf{p}')$ is exactly the momentum derivative of phase shift $\theta(\mathbf{p},\mathbf{p}')$ of the two-body scattering between quasiparticles. So, Eq. (14) can be regarded as a generalization of the TBA equation to that for quasiparticles in Haldane liquid.

As is known in the one-dimensional systems such as the CSM, *the TBA equation is exactly equivalent to the Wu's relation for the distribution function* $W_{\mathbf{p}}$ *for an ideal gas with HWS* [11,12]. In our case of the higher-dimensional systems, it also holds true as follows: Supposing $W_{\mathbf{p}} = e^{\beta(\epsilon_{\mathbf{p}}-\mu)}$, Eq. (14) becomes

$$\ln W_{\mathbf{p}} - \sum_{\mathbf{p}'} \phi(\mathbf{p}',\mathbf{p}) \ln(1 + 1/W_{\mathbf{p}'}) = \beta(\epsilon_{\mathbf{p}}^{(0)} - \mu).$$

So, if I put the relation,

$$g(\mathbf{p}',\mathbf{p}) \equiv \delta(\mathbf{p} - \mathbf{p}') + \phi(\mathbf{p}',\mathbf{p}), \tag{15}$$

then I get

$$(1 + W_{\mathbf{p}}) \prod_{\mathbf{p}'} \left(\frac{W_{\mathbf{p}'}}{1 + W_{\mathbf{p}'}} \right)^{g(\mathbf{p}',\mathbf{p})} = e^{\beta(\epsilon_{\mathbf{p}}^{(0)}-\mu)}. \tag{16}$$

This is exactly the condition that was recently discussed by the author [15] and that Bernard and Wu [11] have conjectured for an ideal gas with HWS in arbitrary dimensions. Thus, the Landau's quasiparticle picture also provides a good foundation for the quasiparticles in a Haldane liquid as well, providing a microscopic origin for the HWS.

Let us consider the relationship between $-\partial \mathbf{f}(\mathbf{p}',\mathbf{p})/\partial \mathbf{p}$ in Eq. (10) and $\phi(\mathbf{p},\mathbf{p}')$ in Eq. (15). Bernard and Wu [11] suggested that they are related to each other where $V(\mathbf{p},\mathbf{p}')$ and $\phi(\mathbf{p},\mathbf{p}')/2\pi$ in their notation correspond to $f(\mathbf{p},\mathbf{p}')$ and $\phi(\mathbf{p},\mathbf{p}')$ in our notation, respectively, while reserving the same notation for the mutual statistical parameter $g(\mathbf{p},\mathbf{p}')$ [see Eqs. (40) and (41) in Ref. [11]]. I prove here that they are identical to each other as follows: Suppose that the excitation spectrum has the linear form: $\epsilon_{\mathbf{p}} = v_F|p - p_F|$ in the vicinity of the pseudo-Fermi surface. (This assumption does not mean that $\epsilon_{\mathbf{p}} = \epsilon_{\mathbf{p}}^{(0)}$ since $p \neq p_0$ due to the particle-hole asymmetry.) Using the definition for $\phi(\mathbf{p},\mathbf{p}')$ of Eq. (13) and the symmetry between $\mathbf{p}$ and $\mathbf{p}'$, then

$$\begin{aligned}\phi(\mathbf{p}',\mathbf{p}) &\equiv \partial f(\mathbf{p}',\mathbf{p})/\partial \epsilon_{\mathbf{p}'} = -\partial f(\mathbf{p}',\mathbf{p})/\partial \epsilon_{\mathbf{p}} \\ &= -(1/v_F)\partial f(\mathbf{p}',\mathbf{p})/\partial p = -(m/p_F)\partial f(\mathbf{p}',\mathbf{p})/\partial p \\ &\simeq -m\partial^2 f(\mathbf{p}',\mathbf{p})/\partial p\, \partial p = -m\partial^2 f(\mathbf{p}',\mathbf{p})/\partial \mathbf{p}\, \partial \mathbf{p}.\end{aligned}$$

Hence, I obtain the following theorem:

Theorem 1:

$$\phi(\mathbf{p}',\mathbf{p}) \equiv \frac{\partial f(\mathbf{p}',\mathbf{p})}{\partial \epsilon_{\mathbf{p}'}} = -m \frac{\partial^2 f(\mathbf{p}',\mathbf{p})}{\partial \mathbf{p}\, \partial \mathbf{p}} = -\frac{\partial \mathbf{f}(\mathbf{p}',\mathbf{p})}{\partial \mathbf{p}}. \tag{17}$$

This guarantees the exact relationship between the dynamical interaction $f(\mathbf{p},\mathbf{p}')$ and the mutual statistics $g(\mathbf{p},\mathbf{p}')$ although Bernard and Wu [11] were not able to guarantee this relationship.

Let us now generalize the Luttinger's theorem [3] for a Fermi liquid to that for a Haldane liquid. To do so, let us first evaluate the pseudo-Fermi sphere $v_F^{(0)}$ for a Haldane gas. Suppose that the total number of free quasiparticles is fixed as N. At zero temperature where $\mu = \epsilon_F^{(0)}$, the HW distribution function has the following property: $n_{\mathbf{p}} = 1/g(= 0)$ for $\epsilon \leq \epsilon_F^{(0)}(\epsilon > \epsilon_F^{(0)})$, which means the *particle-hole asymmetry* [Eq. (9)] as a consequence of the *particle-hole duality* [11–13] in the system of quasiparticles in a Haldane liquid at very low temperature. Substituting this into $N = \sum_{\mathbf{p}} n_{\mathbf{p}}$, I obtain

$$N = \frac{1}{g} \sum_{\mathbf{p}} \theta(\epsilon_F^{(0)} - \epsilon_{\mathbf{p}}^{(0)}) = \frac{1}{g} \frac{V}{(2\pi\hbar)^D} V_F^{(0)},$$

where $\theta(\epsilon)$ is a step function and the volume of the pseudo-Fermi sphere for a Haldane gas is denoted by $V_F^{(0)}$, which is given as

$$\begin{aligned} V_F^{(0)} &= \int_{|p| \leq p_{F,0}} d^D p = \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_{0 \leq p \leq p_{F,0}} p^{D-1}\, dp \\ &= \frac{\pi^{D/2}}{\Gamma(D/2 + 1)} p_{F,0}^D \end{aligned}$$

with $\Gamma(s)$ the gamma function. Solving the above for $p_{F,0}$, I obtain the pseudo-Fermi momentum

$$p_{F,0} = \left[g \frac{(2\pi\hbar)^D}{\pi^{D/2}} \Gamma(D/2 + 1) d \right]^{1/D}$$

where $d = N/V$. Hence the pseudo-Fermi energy $\epsilon_F^{(0)}$ is given by $\epsilon_F^{(0)} = p_{F,0}^2/2m$.

Let us next consider the pseudo-Fermi sphere V_F for a Haldane liquid. Following the argument of Luttinger [3], define $f(\epsilon) = \epsilon - \epsilon_{\mathbf{p}}^{(0)} - K_{\mathbf{p}}(\epsilon)$, where $K_{\mathbf{p}}(\epsilon)$ is the self-energy part of the renormalization equation [3] such as Dyson's equation [20] and Eq. (11) in the mean-field picture. This has a discontinuity at the pseudo-Fermi surface $\epsilon = \epsilon_F$ such that $f(\epsilon) > 0$ (< 0) when $\epsilon < \epsilon_F$ $(> \epsilon_F)$. Therefore, the total number N is given by

$$N = \frac{1}{g} \sum_{\mathbf{p}} \theta[\epsilon_F - \epsilon_{\mathbf{p}}^{(0)} - K_{\mathbf{p}}(\epsilon_F)] = \frac{1}{g} \frac{V}{(2\pi\hbar)^D} V_F.$$

Since the total number N of the interacting quasiparticles are kept at the same as that of the free quasiparticles, I conclude the following theorem:

Theorem 2:

$$V_F = V_F^{(0)},$$

and hence

$$p_F = p_{F,0}. \tag{18}$$

This theorem means that the volume of the pseudo-Fermi sphere for an ideal Haldane gas is conserved depending on the statistical parameter g with degeneracy $1/g$ under the introduction of the interaction between quasiparticles in a Haldane liquid. The volume of the pseudo-Fermi sphere coincides with that of the Fermi sphere with degeneracy unity when $g = 1$. On the other hand, it vanishes as $g \to 0$ with an infinite degeneracy so that one can interpret that *the BE condensation is a condensation of all quasiparticles with BES to the pseudo-Fermi sphere with an infinitesimal volume in momentum space.* And therefore, the true BE condensation occurs only when the pure boson case of $g = 0$ as was recently proved by the author [15]. Thus, the above theorem can be thought of as a generalization of the Luttinger's theorem for a Fermi liquid to that for a Haldane liquid.

Let us consider the specific heat of a Haldane liquid. As we studied by Nayak and Wilczek [13], Isakov, Arovas, Myrheim, and Polychronakos [14], and the author [21], the specific heat of a Haldane gas is T linear at very low temperature unless $g = 0$. Here, the author [21] has presented the most complete expression for the specific heat C_v in terms of the language of the generalized Riemann zeta function $\zeta_g(s)$:

$$C_v = \gamma_g T + O(T^2), \tag{19}$$

$$\gamma_g = 2a_g(2)k_B^2 N_D(\epsilon_F), \qquad a_g(2) = \zeta_g(2) + \zeta_{1/g}(2), \tag{20}$$

where $N_D(\epsilon)$ is the density of states for noninteracting quasiparticles in the D-dimensional lattice system and the generalized Riemann zeta function $\zeta_g(s)$ is defined as

$$\zeta_g(s) = \sum_{l=1}^{\infty} \frac{c_l(g)}{l^s},$$

$$c_l(g) = \frac{(-1)^{l+1}}{lg} \frac{[lg]!}{(l-1)![l(g-1)]!}. \tag{21}$$

Here the coefficients $c_l(g)$ are those of the Sutherland expansion (i.e., the cluster expansion) for the CSM in one dimension [7], and appeared even in higher dimensional systems with HWS [10,14,15].

This is a consequence of the existence of the pseudo-Fermi surface: Each of the Nk_BT/ϵ_F quasiparticles has a thermal energy of the order of k_BT unless $g = 0$. Therefore, the total thermal kinetic energy E is of the order of $E \simeq (Nk_BT/\epsilon_F)k_BT$. Hence, the specific heat c_v is given by $c_v \simeq (Nk_B^2/\epsilon_F)T$. The above T-linear specific heat holds true even for a Haldane liquid as well. Because if statistics g is taken care of by the interacting quasiparticle picture and even if the spectrum of the quasiparticle is changed to distort only the density of states of the system, the above argument works for any statistical parameter $g > 0$. And other quantities such as magnetic susceptibility and compressibility would be obtained by a similar way as was so in the Landau's Fermi liquid theory [1].

In conclusion, I have discussed the foundation of a liquid with HWS—a Haldane liquid—in arbitrary dimensions, as generalizing the point of view of the Landau's Fermi liquid theory [1]. I have shown that many properties of a Haldane liquid, if it exists in nature, are shared with those of a Luttinger liquid [6] in one dimension. It is very interesting to study what a physical system satisfies the Haldane liquid properties.

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