Correlations in Chaotic Eigenfunctions at Large Separation

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An energy eigenfunction in a classically chaotic system is known to have spatial correlations which (in the limit of small $\hbar$) are governed by a microcanonical distribution in the classical phase space. This result is valid, however, only over coordinate distances which are small compared to any relevant classical distance scales (such as the cyclotron radius for a charged particle in a magnetic field). We derive a modified formula for the correlation function in the regime of large separation. This then permits a complete description, over all length scales, of the statistical properties of chaotic eigenfunctions in the $\hbar \to 0$ limit. Applications to quantum dots are briefly discussed. [S0031-9007(98)05350-2]

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In a Hamiltonian system which exhibits classical chaos throughout the accessible phase space, the quantum energy eigenvalues and eigenfunctions are known to have certain universal properties in the limit of small $\hbar$ (which, in practice, is achieved at sufficiently high energy) [1,2]. We will be interested in the energy eigenfunctions; these can be characterized as random variables with a Gaussian probability distribution of the form [3–11]

$$P(\psi \,|\, E) \propto \exp\Bigg[-\frac{\beta}{2} \int d^f q_1 \times \int d^f q_2 \, \psi^*(\mathbf{q}_2) K(\mathbf{q}_2, \mathbf{q}_1 \,|\, E) \psi(\mathbf{q}_1) \Bigg], \tag{1}$$

where f is the number of degrees of freedom. If the system is time-reversal invariant, the eigenfunctions are real and $\beta = 1$; if it is not, they are complex and $\beta = 2$. The measure corresponding to Eq. (1) is the standard one of Euclidean quantum field theory: $\mathcal{D}\psi = \prod_{\mathbf{q}} d\psi(\mathbf{q})$ for $\beta = 1$ and $\mathcal{D}\psi = \prod_{\mathbf{q}} d\,\mathrm{Re}\,\psi(\mathbf{q}) d\,\mathrm{Im}\,\psi(\mathbf{q})$ for $\beta = 2$. $P(\psi \,|\, E)\mathcal{D}\psi$ represents the probability that the actual eigenfunction $\psi_\alpha(\mathbf{q})$ for $E = E_\alpha$ (a particular energy eigenvalue) is between $\psi(\mathbf{q})$ and $\psi(\mathbf{q}) + d\psi(\mathbf{q})$ for all coordinates $\mathbf{q}$. The kernel $K(\mathbf{q}_2, \mathbf{q}_1 \,|\, E)$ is the functional inverse of the two-point correlation function

$$C(\mathbf{q}_2, \mathbf{q}_1 \,|\, E) \equiv \int \psi(\mathbf{q}_2) \psi^*(\mathbf{q}_1) P(\psi \,|\, E) \, \mathcal{D}\psi \,. \tag{2}$$

The explicit formula for $C(\mathbf{q}_2, \mathbf{q}_1 \,|\, E)$ which was originally suggested by Berry [3] assumes a microcanonical probability density in the classical phase space,

$$C(\mathbf{q}_2, \mathbf{q}_1 \,|\, E) = \frac{1}{\bar{\rho}(E)} \int \frac{d^f p}{(2\pi\hbar)^f} \, e^{i\mathbf{p}\cdot(\mathbf{q}_2 - \mathbf{q}_1)/\hbar} \times \delta[E - H_W(\mathbf{p}, \bar{\mathbf{q}})], \tag{3}$$

where $\bar{\mathbf{q}} \equiv \frac{1}{2}(\mathbf{q}_1 + \mathbf{q}_2)$, $H_W(\mathbf{p}, \mathbf{q})$ is the classical Hamiltonian (more specifically, it is the Weyl symbol of the Hamiltonian operator), and $\bar{\rho}(E)$ is the semiclassical density of states,

$$\bar{\rho}(E) = \int \frac{d^f p \, d^f q}{(2\pi\hbar)^f} \, \delta[E - H_W(\mathbf{p}, \mathbf{q})]. \tag{4}$$

However, Eq. (3) applies only when the separation $|\mathbf{q}_2 - \mathbf{q}_1|$ is sufficiently small [3]. Consider, for example, the case $H = \mathbf{p}^2/2m + V(\mathbf{q})$; we would not expect the correlations in an eigenfunction at two points $\mathbf{q}_1$ and $\mathbf{q}_2$ to depend only on the value of the potential at $\bar{\mathbf{q}}$ if $V(\bar{\mathbf{q}})$ has a significantly different value than either $V(\mathbf{q}_1)$ or $V(\mathbf{q}_2)$. Our goal, then, is to find the correct formula for $C(\mathbf{q}_2, \mathbf{q}_1 \,|\, E)$ when $|\mathbf{q}_2 - \mathbf{q}_1|$ is large.

One way to motivate Eq. (3) is to consider the energy Green's function [12,13]

$$G(\mathbf{q}_2, \mathbf{q}_1 \,|\, E) \equiv \sum_\alpha \frac{\psi_\alpha(\mathbf{q}_2) \psi_\alpha^*(\mathbf{q}_1)}{E - E_\alpha + i\epsilon}, \tag{5}$$

where $\epsilon \to 0^+$. We then have

$$\sum_\alpha \psi_\alpha(\mathbf{q}_2) \psi_\alpha^*(\mathbf{q}_1) \delta(E - E_\alpha) = \frac{1}{2\pi i} [G(\mathbf{q}_1, \mathbf{q}_2 \,|\, E)^* - G(\mathbf{q}_2, \mathbf{q}_1 \,|\, E)], \tag{6}$$

and the exact density of states is

$$\rho(E) \equiv \sum_\alpha \delta(E - E_\alpha) \tag{7}$$

$$= \frac{1}{2\pi i} \int d^f q \, [G(\mathbf{q}, \mathbf{q} \,|\, E)^* - G(\mathbf{q}, \mathbf{q} \,|\, E)]. \tag{8}$$

While the exact Green's function clearly has singularities whenever $E = E_\alpha$, its leading approximation $\bar{G}(\mathbf{q}_2, \mathbf{q}_1 \,|\, E)$ in the small-$\hbar$ limit is a smooth function of its arguments. Given this, Eqs. (2), (6), and (7) make it natural to expect that

$$C(\mathbf{q}_2, \mathbf{q}_1 \,|\, E) = \frac{1}{2\pi i \bar{\rho}(E)} \times [\bar{G}(\mathbf{q}_1, \mathbf{q}_2 \,|\, E)^* - \bar{G}(\mathbf{q}_2, \mathbf{q}_1 \,|\, E)]. \tag{9}$$

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This formula can be derived explicitly [6] in the theory of disordered metals, where a white-noise random potential is added to the Hamiltonian. In this case, Eq. (9) holds with $\bar{G}$ standing for the Green's function averaged over the random potential. In the limit that the strength of the potential is large (which corresponds to the $\hbar \to 0$ limit for chaotic systems), Eq. (1) for the eigenfunction probability can also be derived explicitly [5–8,10]. Corrections for finite potential strength (that is, finite $\hbar$) can also be computed [14], and are complimentary to the calculations done here.

Let us now briefly recall the construction of $\bar{G}(\mathbf{q}_2, \mathbf{q}_1 \,|\, E)$ in the $\hbar \to 0$ limit [12,13]. The energy Green's function is related to the propagator $\langle \mathbf{q}_2 | e^{-iHt/\hbar} | \mathbf{q}_1 \rangle$ via

$$G(\mathbf{q}_2, \mathbf{q}_1 \,|\, E) = \frac{1}{i\hbar} \int_0^\infty dt\, e^{i(E+i\epsilon)t/\hbar} \langle \mathbf{q}_2 | e^{-iHt/\hbar} | \mathbf{q}_1 \rangle \,. \tag{10}$$

The propagator can be written as

$$\langle \mathbf{q}_2 | e^{-iHt/\hbar} | \mathbf{q}_1 \rangle = \int \frac{d^f p}{(2\pi\hbar)^f} \, e^{i\mathbf{p}\cdot(\mathbf{q}_2 - \mathbf{q}_1)/\hbar} \times (e^{-iHt/\hbar})_W(\mathbf{p}, \bar{\mathbf{q}}) \,, \tag{11}$$

where again $\bar{\mathbf{q}} = \frac{1}{2}(\mathbf{q}_1 + \mathbf{q}_2)$, and $A_W(\mathbf{p}, \bar{\mathbf{q}})$ denotes the Weyl symbol of the operator A; in fact, Eq. (11) is simply the Fourier transform of the definition of the Weyl symbol. For sufficiently small times,

$$(e^{-iHt/\hbar})_W(\mathbf{p}, \mathbf{q}) \simeq e^{-iH_W(\mathbf{p},\mathbf{q})t/\hbar} . \tag{12}$$

Inserting Eqs. (11) and (12) into Eq. (10), and performing the time integral, we get the desired approximation,

$$\bar{G}(\mathbf{q}_2, \mathbf{q}_1 \,|\, E) = \int \frac{d^f p}{(2\pi\hbar)^f} \, e^{i\mathbf{p}\cdot(\mathbf{q}_2 - \mathbf{q}_1)/\hbar} \times \frac{1}{E - H_W(\mathbf{p}, \bar{\mathbf{q}}) + i\epsilon} \,. \tag{13}$$

The derivation of this formula is flawed, however, since we integrated over all positive times even though Eq. (12) is valid only for short times. A more careful analysis shows that Eq. (13) is valid in the limit of small $\hbar$, provided that $|\mathbf{q}_2 - \mathbf{q}_1|$ is small enough so that the shortest classical path $\mathbf{q}(t)$ connecting $\mathbf{q}_1$ to $\mathbf{q}_2$ with energy E is well approximated by a linear function of time. Note that this criterion is purely classical; $|\mathbf{q}_2 - \mathbf{q}_1|$ can be sufficiently small even if it is large compared with the quantum wavelength $\hbar/p$, where $p = |\mathbf{p}|$ is the magnitude of the classical momentum when $\mathbf{q} = \bar{\mathbf{q}}$. With this caveat, Eq. (13), when inserted into Eq. (9), immediately yields Eq. (3).

As this derivation shows, however, Eq. (3) is not valid if $|\mathbf{q}_2 - \mathbf{q}_1|$ is too large. In this case, we must use a different semiclassical formula for $G(\mathbf{q}_2, \mathbf{q}_1 \,|\, E)$, making a stationary phase approximation both in the Feynman path integral representation of the propagator, and in the time integral of Eq. (10). The well-known result is [12]

$$\bar{G}(\mathbf{q}_2, \mathbf{q}_1 \,|\, E) = \frac{1}{i\hbar(2\pi i\hbar)^{(f-1)/2}} \times \sum_{\text{paths}} |D_{\text{p}}|^{1/2} e^{iS_{\text{p}}/\hbar - i\nu_{\text{p}}\pi/2} . \tag{14}$$

Here the sum is over all classical paths connecting $\mathbf{q}_1$ to $\mathbf{q}_2$ with energy E and action

$$S_{\text{p}} = \int_{\mathbf{q}_1}^{\mathbf{q}_2} \mathbf{p} \cdot d\mathbf{q} \,. \tag{15}$$

The index ν_{p} counts the number of classical focal points along the path, and the determinant D_{p} of second derivatives of S_{p} is given by

$$D_{\text{p}} = \det \begin{pmatrix} \frac{\partial^2 S_{\text{p}}}{\partial \mathbf{q}_2 \partial \mathbf{q}_1} & \frac{\partial^2 S_{\text{p}}}{\partial E \partial \mathbf{q}_1} \\ \frac{\partial^2 S_{\text{p}}}{\partial \mathbf{q}_2 \partial E} & \frac{\partial^2 S_{\text{p}}}{\partial E^2} \end{pmatrix} . \tag{16}$$

It is Eq. (14) which should be used in Eq. (9) when the shortest classical path from $\mathbf{q}_1$ to $\mathbf{q}_2$ is not (approximately) a linear function of time.

If the system is time-reversal invariant, then the paths from $\mathbf{q}_2$ to $\mathbf{q}_1$ have the same set of values of S_{p}, ν_{p}, and D_{p} as the paths from $\mathbf{q}_1$ to $\mathbf{q}_2$. In this case, we have

$$C(\mathbf{q}_2, \mathbf{q}_1 \,|\, E) = \frac{2}{\bar{\rho}(E)\,(2\pi\hbar)^{(f+1)/2}} \sum_{\text{paths}} |D_{\text{p}}|^{1/2} \times \cos[S_{\text{p}}/\hbar - (2\nu_{\text{p}} + f - 1)\pi/4] \tag{17}$$

instead of Eq. (3) when $|\mathbf{q}_2 - \mathbf{q}_1|$ is large.

To illustrate the differences between Eq. (17) and Eq. (3), let us examine a few special cases. First, we consider an f-dimensional billiard in which the straight-line path from $\mathbf{q}_1$ to $\mathbf{q}_2$ is not blocked. In the interior of the billiard, $H = \mathbf{p}^2/2m$, and Eq. (3) yields [3]

$$C(\mathbf{q}_2, \mathbf{q}_1 \,|\, E) = V^{-1}\Gamma(f/2) \, \frac{J_{(f-2)/2}(kL)}{(kL/2)^{(f-2)/2}} \,, \tag{18}$$

where V is the f-dimensional volume of the billiard, $\hbar k = (2mE)^{1/2}$, $L = |\mathbf{q}_2 - \mathbf{q}_1|$ is the length of the straight-line path, $\Gamma(x)$ is the Euler gamma function, and $J_\nu(x)$ is an ordinary Bessel function. In the case at hand, Eq. (18) is valid even for large L. The reason is that the straight-line path is a linear function of time, and this is the condition needed for the validity of Eq. (3). Turning to Eq. (17), we note that $\bar{\rho}(E) = k^f V/(4\pi)^{f/2}\Gamma(f/2)E$, and that the straight-line path has $S_{\text{p}}/\hbar = kL$ and $|D_{\text{p}}| = m^2(2mE)^{(f-3)/2}/L^{f-1}$. If $\mathbf{q}_1$ and $\mathbf{q}_2$ are both far from any of the billiard's walls, the straight-line path makes the

dominant contribution, and we find

$$C(\mathbf{q}_2, \mathbf{q}_1 \,|\, E) = V^{-1}\Gamma(f/2) \times \frac{\cos[kL - (2\nu_p + f - 1)\pi/4]}{\pi^{1/2}(kL/2)^{(f-1)/2}}, \tag{19}$$

with $\nu_p = 0$. Equation (19) is equivalent to Eq. (18) when kL is large, since in this regime the asymptotic form of the Bessel function can be invoked. Thus, in the present case, both Eq. (3) and Eq. (17) are valid for large L, and Eq. (3) is valid for small L (less than a quantum wavelength, $2\pi/k$) as well.

On the other hand, if we consider a billiard in which the straight-line path from $\mathbf{q}_1$ to $\mathbf{q}_2$ is blocked by an obstacle, such as in Fig. 1, Eq. (18) is not correct, and we must use Eq. (17). The shortest classical path connecting $\mathbf{q}_1$ to $\mathbf{q}_2$ makes the dominant contribution; this is given by Eq. (19), provided we take L to be the length of the path, and set ν_p equal to twice the number of bounces. In most realistic cases of this type, there will be many other paths with more bounces that are not very much longer; these will all contribute to $C(\mathbf{q}_2, \mathbf{q}_1 \,|\, E)$ as well. However, if we attempt to verify Eq. (19) numerically for a particular eigenfunction of a particular billiard, it is necessary to average over a range of $\mathbf{q}_1$ and $\mathbf{q}_2$ in order to reduce the variance in $C(\mathbf{q}_2, \mathbf{q}_1 \,|\, E)$ which is expected from the probability distribution (1) [3,4,11]. If this averaging is carried out with the length L of the shortest classical path held fixed, the other contributing paths will in general have lengths that vary. If this variation is large on the scale of the quantum wavelength $2\pi/k$, then the net contribution of all these other paths to the averaged $C(\mathbf{q}_2, \mathbf{q}_1 \,|\, E)$ should be small, rendering Eq. (19) a valid formula for the averaged correlation function.

As another example we consider a billiard with the addition of an isotropic harmonic potential $V(\mathbf{q}) = \frac{1}{2}m\omega^2\mathbf{q}^2$. This case is of some physical interest for $f = 2$; wave-function correlations in quantum dots have been studied assuming that the dot is well modeled by a two-dimensional billiard, but in fact there is also a smooth confining potential, often approximated as harmonic. For simplicity, we give only the leading corrections in the limit of a weak potential. The shortest classical path (assuming it is not blocked) has action

$$S_p/\hbar = kL\left[1 + \frac{L^2 - 3d^2}{12R^2} + \ldots\right], \tag{20}$$

where $\hbar k = (2mE)^{1/2}$ and $L = |\mathbf{q}_2 - \mathbf{q}_1|$ as before, and we have introduced $d^2 \equiv \mathbf{q}_1^2 + \mathbf{q}_2^2$ and $R \equiv (2E/m\omega^2)^{1/2}$; R is the maximum distance from the origin which can be reached with energy E. Also, we find

$$|D_p| = |D_p|_{\omega=0}\left[1 + \frac{(f-1)L^2 - (f-3)d^2}{4R^2} + \ldots\right]. \tag{21}$$

In all but two dimensions, the correction to $|D_p|$ is dominated by the correction to $\bar{\rho}(E)$, which is $O(V^{2/f}/R^2)$, where $V^{1/f}$ is the linear size of the billiard. However, when $f = 2$, $\bar{\rho}(E)$ is independent of E, and it is not changed by the presence of a weak potential. In this case, Eq. (21) represents the dominant correction to the amplitude of $C(\mathbf{q}_2, \mathbf{q}_1 \,|\, E)$.

As a final example with importance for quantum dots, we consider a particle with charge e in a two-dimensional billiard with a uniform, perpendicular magnetic field $\mathbf{B}$. In the billiard interior we have $H = (\mathbf{p} - e\mathbf{A})^2/2m$, and we will work in the gauge in which the vector potential is $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{q}$. This system is not time-reversal invariant, and so we must use Eqs. (9) and (14) rather than Eq. (17). We again consider points $\mathbf{q}_1$ and $\mathbf{q}_2$ which are far from the billiard's walls. The shortest classical path is then a circular arc with length ℓ, related to the separation $L = |\mathbf{q}_2 - \mathbf{q}_1|$ and classical cyclotron radius $R \equiv (2mE)^{1/2}/|eB|$ via $\ell = 2R\sin^{-1}(L/2R)$. The action for this path can be divided into a geometric part and a gauge-dependent part, $S_p = S_{geom} + S_{gauge}$. The geometric part is

$$S_{geom} = \hbar k\left(\ell - \frac{\mathcal{A}}{R}\right) = \hbar k L\left(1 - \frac{L^2}{24R^2} + \ldots\right), \tag{22}$$

where again $\hbar k = (2mE)^{1/2}$, and $\mathcal{A} = \frac{1}{2}R\ell - \frac{1}{2}R^2 \times \sin(\ell/R)$ is the area enclosed by the circular arc and the straight line connecting $\mathbf{q}_1$ to $\mathbf{q}_2$. The gauge-dependent part is energy independent, and changes sign when $\mathbf{q}_1$ and $\mathbf{q}_2$ are exchanged. For our gauge choice,

$$S_{gauge} = \tfrac{1}{2}e\mathbf{B} \cdot (\mathbf{q}_1 \times \mathbf{q}_2). \tag{23}$$

If we make a gauge transformation

$$\mathbf{A}(\mathbf{q}) \rightarrow \mathbf{A}(\mathbf{q}) + \nabla\Phi(\mathbf{q}), \tag{24}$$

where $\Phi(\mathbf{q})$ is any smooth function, then

$$S_{gauge} \rightarrow S_{gauge} + e\Phi(\mathbf{q}_2) - e\Phi(\mathbf{q}_1). \tag{25}$$

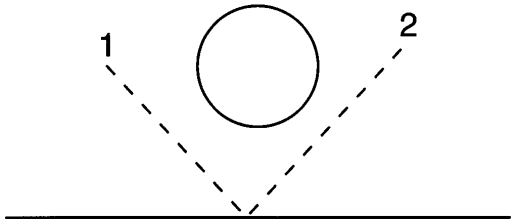

FIG. 1. A portion of a Sinai billiard in which the circular scatterer is off center; the path shown is the shortest classical path between points 1 and 2.

The determinant $|D_{\rm p}|$, on the other hand, is gauge invariant,

$$|D_{\rm p}| = \frac{m^2}{\hbar k L}\left(1 - \frac{L^2}{4R^2}\right)^{-1/2}. \quad (26)$$

Again there is no correction to $\bar{\rho}(E)$ in two dimensions. Keeping only the contribution of this path, we find from Eqs. (9) and (14) that

$$C(\mathbf{q}_2, \mathbf{q}_1 \,|\, E) = V^{-1} \exp(iS_{\rm gauge}/\hbar) \times \frac{\cos(S_{\rm geom}/\hbar - \pi/4)}{(\pi k L/2)^{1/2}(1 - L^2/4R^2)^{1/4}}, \quad (27)$$

where V is the area of the billiard. Under the gauge transformation (24), Eq. (25) implies

$$C(\mathbf{q}_2, \mathbf{q}_1 \,|\, E) \to e^{+ie[\Phi(\mathbf{q}_2)-\Phi(\mathbf{q}_1)]/\hbar} C(\mathbf{q}_2, \mathbf{q}_1 \,|\, E). \quad (28)$$

That this is correct can be seen by recalling that a wave function $\psi(\mathbf{q})$ transforms as

$$\psi(\mathbf{q}) \to e^{+ie\Phi(\mathbf{q})/\hbar}\psi(\mathbf{q}) \quad (29)$$

under (24), and that $C(\mathbf{q}_2, \mathbf{q}_1 \,|\, E_\alpha)$ is the expected value of $\psi_\alpha(\mathbf{q}_2)\psi_\alpha^*(\mathbf{q}_1)$. On the other hand, Eq. (3) implies that

$$C(\mathbf{q}_2, \mathbf{q}_1 \,|\, E) \to e^{+ie(\mathbf{q}_2-\mathbf{q}_1)\cdot\nabla\Phi(\bar{\mathbf{q}})/\hbar} C(\mathbf{q}_2, \mathbf{q}_1 \,|\, E), \quad (30)$$

which again illustrates the fact that Eq. (3) is valid only when $|\mathbf{q}_2 - \mathbf{q}_1|$ is sufficiently small.

Finally, we note that our expression for $C(\mathbf{q}_2, \mathbf{q}_1 \,|\, E)$ is needed to resolve a discrepancy between two different formulas in the literature for the expected values of off-diagonal matrix elements (in the energy-eigenstate basis) of simple, $\hbar$-independent operators in classically chaotic systems. This will be the subject of a separate paper.

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[1] *Les Houches LII, Chaos and Quantum Physics*, edited by M.-J. Giannoni, A. Voros, and J. Zinn-Justin (North-Holland, Amsterdam, 1991).
[2] *Quantum Chaos*, edited by G. Casati and B. Chirikov (Cambridge University Press, New York, 1995).
[3] M. V. Berry, J. Phys. A **10**, 2083 (1977).
[4] P. O'Connor, J. Gehlen, and E. J. Heller, Phys. Rev. Lett. **58**, 1296 (1987).
[5] V. N. Prigodin, K. B. Efetov, and S. Idia, Phys. Rev. Lett. **71**, 1230 (1993).
[6] V. N. Prigodin, B. L. Altshuler, K. B. Efetov, and S. Idia, Phys. Rev. Lett. **72**, 546 (1994).
[7] V. N. Prigodin, Phys. Rev. Lett. **74**, 1566 (1995).
[8] V. N. Prigodin, N. Taniguchi, A. Kudrolli, V. Kidambi, and S. Sridhar, Phys. Rev. Lett. **75**, 2392 (1995).
[9] Y. Alhassid and C. H. Lewenkopf, Phys. Rev. Lett. **75**, 3922 (1995).
[10] M. Srednicki, Phys. Rev. E **54**, 954 (1996).
[11] M. Srednicki and F. Stiernelof, J. Phys. A **29**, 5817 (1996).
[12] M. C. Gutzwiller, J. Math. Phys. (N.Y.) **8**, 1979 (1967).
[13] M. V. Berry and K. E. Mount, Rep. Prog. Phys. **35**, 315 (1972).
[14] Ya. M. Blanter and A. D. Mirlin, Phys. Rev. E **55**, 6514 (1997).