Size Scaling of Strength in Heterogeneous Materials

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The strength of planar arrays of n elastic elements, having heterogeneous strengths and coupled by elastic stress transfer, is studied by simulation and size-scaling analyses. Failure statistics for large n are shown to be controlled by a critical damage cluster of size n_c whose failure statistics are the same as the *known* statistics for n_c elements failing under mean-field stress transfer. The large n statistics are found analytically. Strength scales as $\sqrt{\ln(n)}$, accurate to $\geq 10^{11}$ elements. These results suggest new concepts for understanding scaling of failure in heterogeneous materials. [S0031-9007(98)05351-4]

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Fracture in heterogeneous systems has been an area of active study in recent years due to its relevance in practical engineering materials such as polycrystals and fiber composites [1], biological materials such as bone [2], and possibly geological systems such as earthquake faults [3]. Elastic stresses are long ranged, however, so that regions of high local damage generate high local stresses which start an avalanche of breaks across the entire macroscopic system. The major challenge in dealing with fracture problems is thus combining the statistical evolution of damage initiating around the weaker heterogeneities with the associated stress redistributions to accurately predict the point of instability. Fracture problems therefore cannot be described by mean-field theories, in general.

A consequence of the localized nature of fracture is that the strength at a fixed size is inherently statistically distributed. Nominally identical materials (same geometry, underlying distribution of element strengths but a different statistical realization) have different strengths. Another consequence of the localized nature of the fracture instability is that the fracture strength is governed by weak-link scaling. That is, a large system can be formally considered as composed of a collection of independent subsystems coupled in series so that failure in the weakest subsystem causes failure across the entire system. So, the strength distributions at different sizes must be related as follows. Let $H_n(\sigma)$ and $H_{n'}(\sigma)$ be the cumulative failure probability distributions (FPD) at stress σ for systems of size n and n', respectively. Considering the size n system as composed of n/n' subsystems of size $n' < n$, the probability of survival at size n, $1 - H_n(\sigma)$, is then simply the product of the probabilities of survival $1 - H_{n'}(\sigma)$ for the n/n' subsystems. The FPD $H_n(\sigma)$ at size n can thus be related to that of size n' by [4–11]

$$H_n(\sigma) = 1 - [1 - H_{n'}(\sigma)]^{n/n'}. \quad (1)$$

Recent work on the heterogeneous fracture problem has drawn some analogies between fracture and first-order phase transitions [12], and also related the scaling of some subcritical damage events to predictions from mean-field theory for special distributions of heterogeneity [13]. While intriguing, such relationships probably cannot be rigorously extended to predict the (size-dependent) failure strength since phase transitions and mean-field theory models do not exhibit size scaling of any features such as the critical field, coexistence line, or spinodal line. Also, the fracture problem does not even have a proper thermodynamic limit since the strength approaches zero as the system size becomes infinite. Earlier works using spring, resistor, and beam models to simulate fracture highlighted the size dependence but were generally unable to predict the observed size dependences except for a particular distribution of heterogeneity for which mean-field theories do seem to apply [14].

The heterogeneous failure problem has been studied in detail by statistical approaches [4–11]. The most widely studied problem is a linear array of elements in which broken elements transfer stress only to the immediate two neighbors. Recursive and asymptotic solutions [5,9] show all the features expected when there is local stress transfer: decreasing strength with increasing size, weak-link scaling, and failure driven by the development of a local cluster of breaks that generates high local stress concentrations and leads to a cascade of breaks across the material. The 1D array and extremely localized nature of the load transfer make this problem of limited practical interest. Analyses of hexagonal arrays with nearest neighbor stress transfer [6] and linear arrays with next nearest neighbor stress transfer [7] have been performed, but are not easily extended to realistic load sharing models.

Here, we study the fracture strength of a typical 3D heterogeneous system (aligned array of elements with a strength distribution) with elastic stress transfer from failed to unfailed elements, and provide a new perspective on the prediction of material strength and its size-scaling behavior. Specifically, we demonstrate that a large system of size n can be viewed as a collection of n/n_c smaller systems of a critical size n_c where the fracture statistics (FPD) at size n_c are essentially identical to those of the same size n_c system failing within a mean-field model (no local stress transfer). In other words, at size n_c the dominant fluctuations in the "local-field" (elastic stress transfer) and mean-field problems are the same. The size n_c of the critical region is nonuniversal, depending on the element strength distribution and nature of the load sharing. But, the local-field–mean-field correspondence appears universal and gives rise to a universal form for the size scaling of the strength, which depends on $\sqrt{\ln(n)}$, and to quantitatively accurate predictions for strength. These results suggest that future work on failure in heterogeneous systems should focus on understanding how the fluctuations in the local-field and mean-field problems can be identical at a critical size n_c and on the prediction of the critical size.

To begin, we first discuss the mean-field theory for strength [15]. Consider a very large array of n elements in a plane with stress applied perpendicular to the plane. Let the FPD versus stress σ for the elements follow the Weibull form $P_f(\sigma) = 1 - e^{-\sigma^m}$, where m is the "Weibull modulus" and stress is in units of the characteristic (0.632 probability) strength. In mean-field theory [16], when a stress σ is applied to the array some elements fail and the unbroken elements all carry a stress $T > \sigma$. The fraction of surviving elements is $1 - P_f(T)$ and, since the average stress is σ, one obtains the relationship $\sigma = Te^{-T^m}$. Maximizing σ vs T gives the strength of $\mu_\infty = (me)^{-1/m}$. For a finite-size array ($n > 10$), the failure probability distribution is a Gaussian with cumulative probability denoted as $\Phi_n(z), z = (\sigma - \mu_n)/\gamma_n$, with a mean strength μ_n and a standard deviation γ_n [10,15,16]

$$\mu_n = \mu_\infty + 0.9962 n^{-2/3} m^{-1/3-1/m} e^{-1/m}; \quad (2)$$

$$\gamma_n = \tilde{\gamma}_n \left[1 - 0.317 \left(\frac{\mu_n}{\tilde{\gamma}_n} \right)^2 m^{-2/3} e^{4/3m} n^{-4/3} \right]; \quad (3)$$

$$\tilde{\gamma}_n = n^{-1/2} \left[\left(\frac{1}{m} \right)^{1/m} e^{-1/m} (1 - e^{-1/m}) \right]^{1/2}.$$

Now consider the problem in which broken elements transfer stress predominantly to *nearby* unbroken elements, a problem more closely related to real materials and fiber composites. Geometry now becomes relevant, and so we place the elements in a square array. Hedgepeth and Van Dyke developed a continuum model for the in-plane stress transfer around a broken element and a method for calculating the stress in the presence of an arbitrary spatial distribution of broken elements [17]. We have also developed a Green's function method with an adjustable stress-transfer function that can be set to exactly the stress transfer of [17] in one limit [18]. In that limit, which we use here, the stress transfer from an isolated broken fiber is 0.1428 to the four near neighbors, 0.053 to the next neighbors, and decays as r^{-3} at large distances. Compact clusters of breaks lead to stress concentrations on the cluster perimeter that increase with increasing cluster size due to the long-range elastic interactions. The solution method for obtaining the stress state in the presence of an arbitrary array of breaks involves inversion of a matrix that has a size equal to the number of breaks, not the total number of elements, and is therefore optimally efficient.

We have utilized the technique sketched above to simulate the evolving damage and failure of square arrays of n stochastic elements as follows. Each element is assigned a strength randomly chosen from the Weibull distribution. A stress is applied to the elements such that exactly one break will occur. The load from this element is transferred to the other elements using the Green's function method. The applied load is then adjusted such that one more element has a stress equal to its assigned strength, and that element is broken and the stresses redistributed to all remaining elements. The sequence of breaking an element, redistributing the loads, and adjusting the load to obtain one more broken element, is carried out many times. The maximum applied stress is the tensile strength of that sample. Simulations have been performed for $m = 3, 5, 10$ (typical of many materials and fibers) and $n = 196, 400, 900, 2500$ (except for $m = 3$). Generally, 500 simulations were performed, with 320 ($m = 3$, $n = 900$), 150 ($m = 10$, $n = 2500$), and 100 ($m = 5$, $n = 2500$) being the exceptions.

The simulated FPDs $H_n(\sigma)$ versus stress at each size n and Weibull modulus m are shown in Fig. 1 in a form such that Gaussians plot as straight lines. Clearly, at fixed m, the strength decreases with increasing size and the distributions are not Gaussian. Figure 2 shows examples of the damage just at the maximum applied stress in a typical 196-element array. The extent of damage increases substantially with decreasing Weibull modulus, and macroscopic failure evolves from a localized, but not compact, cluster of broken elements. Since the simulated FPD is size dependent (Fig. 1), and failure is driven by the unstable propagation of a critical cluster rather smaller than the size n (Fig. 2), weak-link scaling should apply.

We now hypothesize that, at fixed element strength distribution (fixed m), there exists a critical size n_c and associated FPD $H_{n_c}(\sigma)$ to which all distributions $H_n(\sigma)$ at

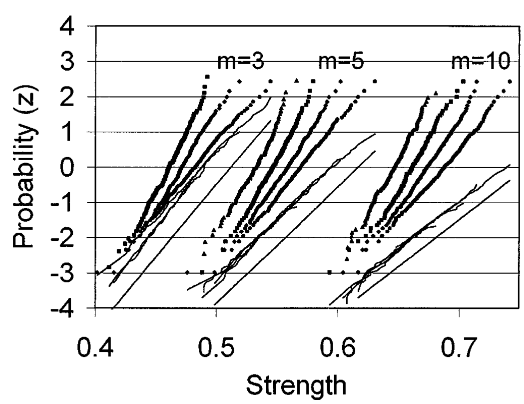

FIG. 1. Cumulative probability of failure versus applied stress, plotted so that Gaussian distributions are straight lines. Simulation data for composite sizes 196 (●), 400 (◆), 900 (■), and 2500 (▲) are shown for $m = 3, 5, 10$. Solid lines are each data set weak-link scaled back to a common Gaussian curve at critical size n_c with $n_c = 30$ ($m = 10$); $n_c = 70$ ($m = 5$); $n_c = 150$ ($m = 3$). The probability of failure for the size n_c in the mean-field problem is shown as the straight solid lines.

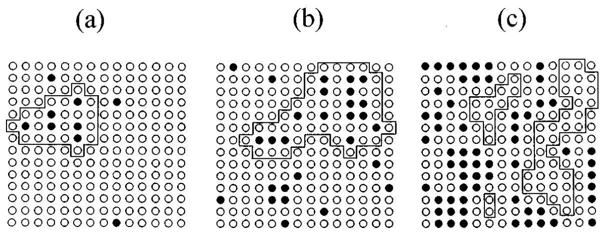

FIG. 2. Examples of fiber damage at the maximum stress in a 196 element array: (a) $m = 10$; (b) $m = 5$; (c) $m = 3$; (broken fibers = ●, unbroken fibers = ○). Estimated critical damage clusters are constructed by using (i) breaks within next nearest neighbor distance of another break in the cluster, (ii) unbroken near neighbor elements of these breaks, and (iii) unbroken elements having two breaks at the next neighbor distance.

larger sizes are related via the weak-link scaling relationship of Eq. (1) (with $n' = n_c$) and that $H_{n_c}(\sigma) = \Phi_{n_c}(z)$, i.e., at the size n_c the FPD is identical to the mean-field FPD. To demonstrate the validity of these postulates, we (i) set $n' = n_c$ in Eq. (1), (ii) use the simulation data for $H_n(\sigma)$ at the various n in Eq. (1), and (iii) search for an n_c for which the derived $H_{n_c}(\sigma)$ is a Gaussian distribution. If the derived Gaussan distribution has the same mean and standard deviation as the mean-field system of the same size, our postulate is demonstrated. Note that the weak-link form of Eq. (1) does not in any way assure that (i) the simulation data versus size will collapse onto a single curve, (ii) the single curve will be Gaussian, or (iii) the Gaussian will have the same mean and standard deviations as the mean-field system of exactly the same size n_c. Such a correspondence is thus highly nontrivial.

Performing the weak-link scaling on each of the simulated probability distributions at sizes $n = 196, 400, 900, 2500$ back to some common size $n' = n_c$, the data for each Weibull modulus are found to collapse well onto a single Gaussian curve, as shown in Fig. 1. The critical sizes are $n_c = 150$ ($m = 3$), $n_c = 70$ ($m = 5$), and $n_c = 30$ ($m = 10$). For each m, the standard deviation of the resulting Gaussian (the inverse slope in Fig. 1) is identical to that for the mean-field problem of exactly the same size n_c. The mean strengths differ by less than 4%, and this has no effect on the predicted scaling with size. Furthermore, the sizes n_c derived purely by scaling are comparable to the critical clusters that cause failure in the simulations (see Fig. 2); n_c can thus be physically interpreted as the critical cluster size driving failure.

The above analysis shows that a large array of n elements fails as if it is composed of a series collection of n/n_c bundles, the weakest of which causes failure. More importantly, the variability in failure strength at size n_c is identical to that in the mean-field system of size n_c, suggesting an intrinsic insensitivity of failure to stress transfer at this size.

Now we obtain an analytic form for the strength distribution. From above, the local-field FPD $H_{n_c}(\sigma)$ at size n_c is given by $\Phi_{n_c}(z)$ [Gaussian with mean $\mu^*_{n_c}$ (not quite μ_{n_c}) and standard deviation γ_{n_c}; see Eqs. (2) and (3)]. Then, we can use the weak-link scaling Eq. (1) and known asymptotic analyses for Gaussian distributions [16,19] to obtain the local-field FPD $H_n(\sigma)$ for arbitrary size n as

$$H_n(\sigma) = 1 - e^{-(\sigma/\tilde{\sigma})^{\tilde{m}}}, \tag{4a}$$

with

$$\tilde{\sigma} = \mu^*_{n_c} - \gamma_{n_c}\sqrt{2\ln(n/n_c)} \times \left[1 - \frac{\ln(\ln(n/n_c)) + \ln(4\pi)}{4\ln(n/n_c)}\right]; \tag{4b}$$

$$\tilde{m} = \frac{\tilde{\sigma}}{\gamma_{n_c}}\sqrt{2\ln(n/n_c)}\,. \tag{4c}$$

The strength follows a Weibull FPD with characteristic strength $\tilde{\sigma}$ and Weibull modulus $\tilde{m}$. The dominant size scaling of the characteristic strength follows $\sqrt{\ln(n)}$, and depends on the standard deviation γ_{n_c} but not on the mean bundle strength $\mu^*_{n_c}$ [both depending on m since $n_c = n_c(m)$]. Therefore, the slight difference in mean strength between local- and mean-field bundles at size n_c ($\mu^*_{n_c}$ vs μ_{n_c}) persists with no change up to large sizes.

The predictions of Eq. (4b) for the characteristic strength $\tilde{\sigma}$ versus size n can be directly compared to the simulation data as follows. Suppose we desire the size n for which the characteristic strength is $\tilde{\sigma}$, i.e., n such that $H_n(\tilde{\sigma}) = 1 - e^{-1}$. Setting the left-hand side of Eq. (1) equal to $1 - e^{-1}$, the size n having strength $\tilde{\sigma}$ must satisfy $n = -n'/\ln[1 - H_{n'}(\tilde{\sigma})]$ where $H_{n'}(\tilde{\sigma})$ at size n' is presumed known. Using the simulation data for n' and $H_{n'}(\tilde{\sigma})$, the strengths obtained in this way and as predicted by Eq. (4b) are shown versus $\sqrt{\ln(n)}$ in Fig. 3. The agreement is excellent mainly because the critical size n_c was determined using the simulations, but this result also demonstrates the accuracy of the local-field–mean-field relationship at size n_c. *Of more importance, these results confirm the quantitative analytic scaling predictions at large sizes, which are*

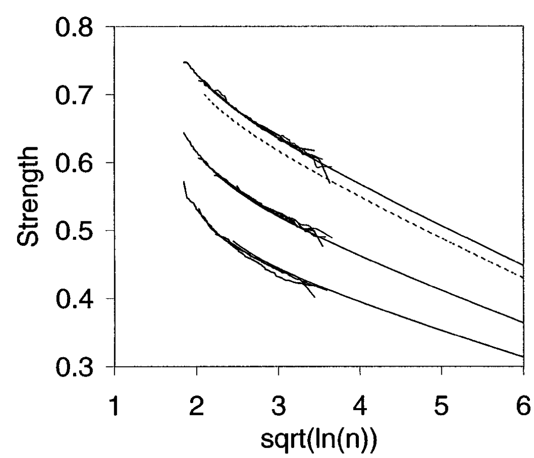

FIG. 3. Simulated and predicted characteristic strength versus size $\sqrt{\ln(n)}$ for $m = 3, 5, 10$. For $m = 10$, predictions use both strengths $\mu^*_{n_c}$ from simulations (solid line) and μ_∞ from the infinite-size mean-field theory (dashed line).

then expected to be quantitatively accurate out to sizes much larger than obtainable by simulation. The present simulation data provides characteristic strengths out to sizes of $\approx 2 \times 10^5$. To obtain the average strength at size n via simulations on size n_s requires $\approx n/n_s$ simulations. Composites commonly tested in laboratory specimens correspond to sizes of at least 10^8 ($\sqrt{\ln(n)} = 4.292$) and actual components can be orders of magnitude larger. The sizes relevant to real materials are thus completely inaccessible by direct simulation. Figure 3 suggests that the strength predictions of the present analytic model will be highly accurate out to $\sqrt{\ln(n)} \approx 4$ ($n \approx 10^7$), and quite good to at least $\sqrt{\ln(n)} \approx 5$ ($n \approx 10^{11}$) and possibly $\sqrt{\ln(n)} \approx 6$ ($n \approx 10^{15}$); the present results thus have significant practical importance to real material systems.

We have shown here that the explicitly size-dependent stochastic phenomenon of failure driven by local stress concentrations is actually controlled by the fluctuations that occur in a mean-field system at a critical size n_c. The general mapping onto a mean-field problem is universal (independent of element statistics for smooth distributions, load sharing [20], and possibly element geometry [21]). Previous associations with a mean-field problem only existed for special element strength distributions [13,14]. We have then found a universal size-scaling behavior of the strength and accurate, quantitative predictions for the strength out to very large, practical, system sizes.

Our results raise the key conceptual question: Why are the fluctuations of the local- (elastic stress transfer) and mean-field problems the same at size n_c? We do not have a definitive answer at present, but offer the following considerations. The sources of fluctuations in both problems are (i) element strength variations, (ii) spatial variations in the break locations, and (iii) fluctuations in the average stress due to variations in the average amount of damage locally. In the local-field problem there are also fluctuations in the element-to-element local stresses. Since the local-field problem at moderate size is controlled by the lower-strength regions of the size n_c mean-field problem, the local occurrence of a cluster of particularly weak fibers may be the main common factor that links the two problems. We also conjecture that, since failure requires some finite amount of damage evolution in a local region, the spatial heterogeneity of the damage homogenizes the stress field over a size comparable to n_c. This reduces the local stress fluctuations that are particular to the local-field problem so that they are not dominant. Finally, the large local-field system is not confined to any *precise* region of space for locating the "critical" region. Rather, the evolving critical damage cluster can wander, searching out the relatively weaker environments. This is deduced from the fact that direct simulations of the size n_c problem do not produce a strength distribution identical to the mean-field distribution, as seen in the data of Fig. 1 for $n = 196$ at $m = 3$ (close to the critical size $n_c = 150$): This is the only data set that does not accurately scale onto the mean-field result. Therefore, limiting the failure to a precise region of size n_c does not give the mean-field fluctuations; the "wandering" of the critical cluster appears important. These general ideas form the skeleton of further analytic work to assess how the local- and mean-field problems can be identical at a critical size and to determine the critical size.

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