Predicting Chaos Most of the Time from Embeddings with Self-Intersections

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Embedding techniques for predicting chaotic time series from experimental data may fail if the reconstructed attractor self-intersects, and such intersections often occur unless the embedding dimension exceeds twice the attractor's box counting dimension. Here we consider embeddings with self-intersection. When the dimension M of the measurement space exceeds the information dimension D_1 of the attractor, reliable prediction is found to be still possible from most orbit points. In particular, the fraction of state space measure from which prediction fails typically scales as ε^{M-D_1} for small ε where ε is the diameter of the current state's neighborhood used for prediction. [S0031-9007(98)05339-3]

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The observed dynamics of many systems cannot be derived from first principles. In some cases, the underlying dynamics display low dimensional behavior, and attempts are made to derive dynamical models from chaotic experimental data [1–3]. Phase space reconstruction techniques have been successfully applied to model low-dimensional dynamics from observed data, revealing important dynamical properties, such as attractor dimensions and Lyapunov exponents, and allowing for short-term prediction of the dynamics [4].

The phase space reconstruction of the system relies on the fact that time series data of generic observables can be used to construct M-dimensional delay-coordinate vectors (forming the so-called measurement space) that correspond *one to one* to phase space states of the system. This result is backed by embedding theorems due to Takens [5] and Sauer *et al.* [6]. The theorems in [6] state that an embedding dimension M larger than twice the box counting dimension D_0 is *sufficient* to fully unfold the attractor in the measurement space for a *generic* observable ($M > 2D_0$). However, favorable choices of observables may unfold the attractor in smaller dimension $M < 2D_0$, while for other observables the same attractor may self-intersect in M-dimensional measurement space.

While some properties of the system, such as the dimensions of the attractor, can be obtained from the measurement whether the attractor is embedded one to one or whether it intersects itself, the dynamics cannot be unambiguously reconstructed at points of self-intersection, and dynamical modeling via embedding fails at these points. For prediction purposes self-intersections of the attractor have therefore been avoided in the past, and techniques have been developed to detect them and to find the required embedding dimension M to avoid them. In particular, the "false nearest neighbor" concept [2,7] has been successfully applied to construct dynamical models in numerous fields, and we note that this concept will be particularly relevant to our discussion in this paper [8].

In this Letter we investigate the quality of the prediction when the attractor is not fully unfolded in the measurement space, but has self-intersections. We find that, provided the dimension M of measurement space is larger than the information dimension D_1 of the underlying dynamics, a prediction based on the reconstructed self-intersecting attractor is possible *most of the time.* More specifically, the measure of points on the attractor, for which the prediction error exceeds any given error bound, vanishes according to the power law $\sim\varepsilon^{M-D_1}$, as the size ε of the neighborhood [9] used for the prediction tends to zero. Therefore, for small enough ε accurate prediction is possible except on a small subset of the attractor even if the attractor is not completely unfolded in measurement space.

This result might justify modeling the dynamics in dimensions $M < 2D_1$ in certain situations where self-intersections occur. For example, this may be advantageous for systems with large attractor dimension, where an embedding in $M > 2D_1$ may not be feasible.

We consider a dynamical system defined on a k-dimensional phase space Y by the map $y_{n+1} = F(y_n)$

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for y_n in Y. We assume that there exists an attractor of the system, whose natural measure is $m(y)$, and whose box counting and information dimension are D_0 and D_1, respectively.

In a typical experiment, the phase space Y may not be observed directly. By constructing time-delay vectors of dimension M from one observable or by measuring M independent observables, the phase space Y is mapped to an M-dimensional *measurement space* X. We denote the mapping from the system's k-dimensional phase space Y to the M dimensional measurement space X by H, and we call H the *measurement function* (cf. Fig. 1). A measurement of the system in the state y in Y yields a measured vector $x = H(y)$ in X. The natural measure $m(y)$ on the attractor projected by H into X space is denoted $\mu(x)$.

Using data from the space X, we attempt to predict the one-step future evolution of the dynamics under F. For this purpose one technique [4,10] is to search the data for previous states x_i in the measurement space time series that lie within a distance ε of the present measured state $x = x_0$ (i.e., those x_i satisfying $\|x_i - x_0\| < \varepsilon$) and average over their known one-step evolution to obtain an estimate $\chi_\varepsilon(x_0)$ for the one-step evolution of the present measured state x_0.

$$\chi_\varepsilon(x_0) = \frac{\sum_{\varepsilon,x_0} H(F(y_i))}{N(\varepsilon, x_0)}. \tag{1}$$

Here $\sum_{\varepsilon,x_0}$ denotes the summation over all points x_i such that $\|x_i - x_0\| < \varepsilon$, and $N(\varepsilon, x_0)$ is the number of such points. For later use we introduce the notation $B_\varepsilon(x_0)$ to stand for the set of x such that $\|x - x_0\| < \varepsilon$. Thus the sum in (1) is over those x_i in the ε ball $B_\varepsilon(x_0)$. As the length of the time series increases, ε can be chosen smaller, and the limiting predicted value $\chi(x_0)$ is then defined by

$$\chi(x_0) = \lim_{\varepsilon \to 0} \chi_\varepsilon(x_0). \tag{2}$$

If the measurement function H is one to one at the point x_0, the prediction $\chi_\varepsilon(x_0)$ becomes more and more accurate as ε goes to zero. In fact, the error in the prediction typically decreases linearly to zero as $\varepsilon \to 0$. If, however, more than one phase space point in the original Y space yields the same measured value x_0 (i.e., more than one phase space point is mapped to the same point x_0), the prediction error at x_0 does not vanish as $\varepsilon \to 0$.

Suppose, for example, that $H(y) = x_0$ has two possible solutions for y, $H(y^{(a)}) = H(y^{(b)}) = x_0$ (cf. Fig. 1). After one time step the points $y^{(a)}$ and $y^{(b)}$ evolve to $F(y^{(a)})$ and $F(y^{(b)})$, respectively (Fig. 1). These two states are generally observed in measurement space X as two different points $x^{(a)} = H(F(y^{(a)}))$ and $x^{(b)} = H(F(y^{(b)}))$. The predicted value $\chi(x_0)$ is now the average of $x^{(a)}$ and $x^{(b)}$ weighted with the appropriate natural measures $m^{(a)}$ and $m^{(b)}$ of the two components of $H^{-1}(B_\varepsilon(x_0))$ near the points $y^{(a)}$ and $y^{(b)}$ (see Fig. 1). If the natural measure at both points is comparable ($m^{(a)} \sim m^{(b)}$), $\chi(x_0)$ lies far from $x^{(a)}$ and $x^{(b)}$, and the prediction is inaccurate. If, however, the natural measures at both points are very different, e.g., $m^{(a)} \gg m^{(b)}$, then $\chi(x_0)$ lies close to $x^{(a)}$, and the prediction is good with probability $m^{(a)}/(m^{(a)} + m^{(b)}) \approx 1$.

Assuming that there are enough time series data of the orbit in X space, so that many points fall in $B_\varepsilon(x_0)$, the average over the time series data points may be replaced by the average with respect to the natural measure $\mu(x)$ on X, i.e.,

$$\chi_\varepsilon(x_0) = \frac{\int_{H^{-1}(B_\varepsilon(x_0))} H(F(y))\, dm(y)}{\mu(B_\varepsilon(x_0))}. \tag{3}$$

This leads us to the following:

Definition 1.—Given a point $x_0 \in X$ with prediction neighborhood $B_\varepsilon(x_0)$ on the image of the attractor, consider

$$\langle(\Delta\chi_\varepsilon)^2\rangle = \frac{\int_{H^{-1}(B_\varepsilon(x_0))}[H(F(y)) - \chi_\varepsilon(x_0)]^2\, dm(y)}{\mu(B_\varepsilon(x_0))}. \tag{4}$$

The finite ε prediction error $\sigma_\varepsilon(x_0)$ and the prediction error $\sigma(x_0)$ at x_0 are defined by

$$\sigma_\varepsilon(x_0) := \sqrt{\langle(\Delta\chi_\varepsilon)^2\rangle}, \quad \text{and} \quad \sigma(x_0) = \lim_{\varepsilon \to 0} \sigma_\varepsilon(x_0), \tag{5}$$

and a point x_0 is said to be *predictable*, if

$$\sigma(x_0) = 0. \tag{6}$$

Conjecture 1.—The natural measure of predictable points is generically one if $M > D_1$.

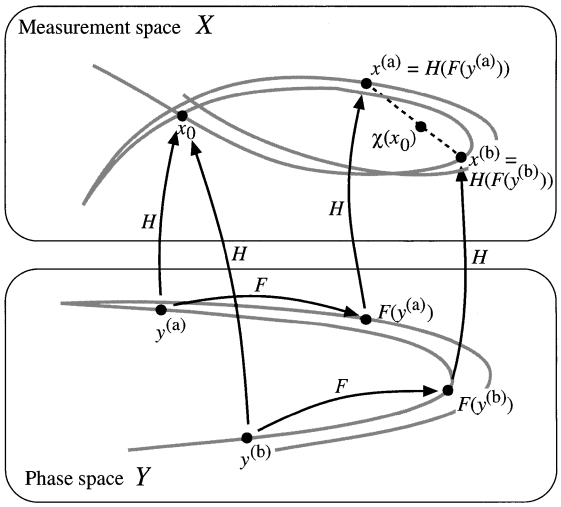

FIG. 1. As observed with the measurement function H, the two points $y^{(a)}$ and $y^{(b)}$ are both recorded as the same point x_0. However, the images of these states under one-step evolution are observed as two different points $x^{(a)} = H(F(y^{(a)}))$ and $x^{(b)} = H(F(y^{(b)}))$. The prediction $\chi(x_0)$ is the weighted average of $x^{(a)}$ and $x^{(b)}$ and lies on the line joining them.

We now give a heuristic argument for Conjecture 1. Given a point on the attractor chosen at random with respect to the natural measure m, the measurement function H can be regarded as placing a distant part of the attractor in X essentially at random with respect to the initially chosen point. The probability p_ε that the chosen point falls in an ε cube with a large measure of another point on the attractor chosen at random with respect to the natural measure is the number of cubes of large measure ($\sim\varepsilon^{-D_1}$) divided by the number of cubes covering measurement space X ($\sim\varepsilon^{-M}$). As $\varepsilon \to 0$ this probability p_ε scales like ε^{M-D_1}.

The finite ε prediction error $\sigma_\varepsilon(x_0)$ is either proportional to ε for small ε if x_0 represents exactly one state in phase space, or else is finite and constant as ε tends to zero [$\sim O(1)$], if x_0 is a point at which parts of the attractor overlap. In the first case, x_0 is predictable; the second case comprises the points that are not predictable. In the second case, for any (not too large) fixed error bound $\delta > 0$, the measure of points with finite prediction error σ larger than δ vanishes like ε^{M-D_1} if $M > D_1$, thus justifying Conjecture 1. This leads directly to the following:

Conjecture 2.—Let $\delta > 0$. The measure $\mu(\sigma_\varepsilon > \delta)$ of points with finite prediction error $\sigma_\varepsilon > \delta$ generically scales in the following way: (i) When $D_1 > M$, then $\mu(\sigma_\varepsilon > \delta) \sim O(1)$. (ii) When $M/2 < D_1 < M$, then $\mu(\sigma_\varepsilon > \delta) \sim \varepsilon^{M-D_1}$, if attractor self-intersections occur [if attractor self-intersections are absent, $\mu(\sigma_\varepsilon > \delta) = 0$]. (iii) If $D_1 < M/2$, then $\mu(\sigma_\varepsilon > \delta) = 0$.

For $D_1 < M/2$, typically no boxes containing large measure overlap. The number of boxes that contain a large measure scales like ε^{-D_1}, such that the number of intersecting boxes is

$$N_x \sim \varepsilon^{-D_1} \times p_\varepsilon \sim \varepsilon^{M-2D_1}, \tag{7}$$

which becomes less than one for $M/2 > D_1$ and ε small enough.

We have numerically verified the above conjectures for the Hénon [11] and the Ikeda map [12]. For the Hénon map we used standard parameters $a = 1.4$ and $b = 0.3$ and two-dimensional time delays ($M = 2$) of the observable $f(x, y) = x + 1.2\sin[5(x - 2y)]$. The observable was chosen because its time delay displays numerous

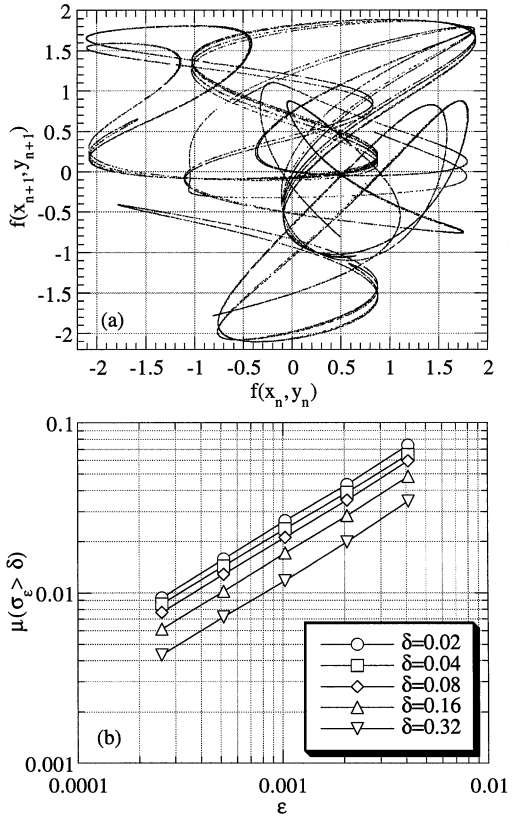

FIG. 2. (a) Hénon map ($a = 1.4, b = 0.3$) as observed with time delay coordinates of the observable f. (b) Scaling of the measure $\mu(\sigma_\varepsilon > \delta)$ of points with prediction error larger than $\delta > 0$. The scaling exponent 0.74 is independent of δ and in good agreement with the exponent $M - D_1 = 2 - 1.26 = 0.74$ given in Conjecture 2. (10^8 time series points were used.)

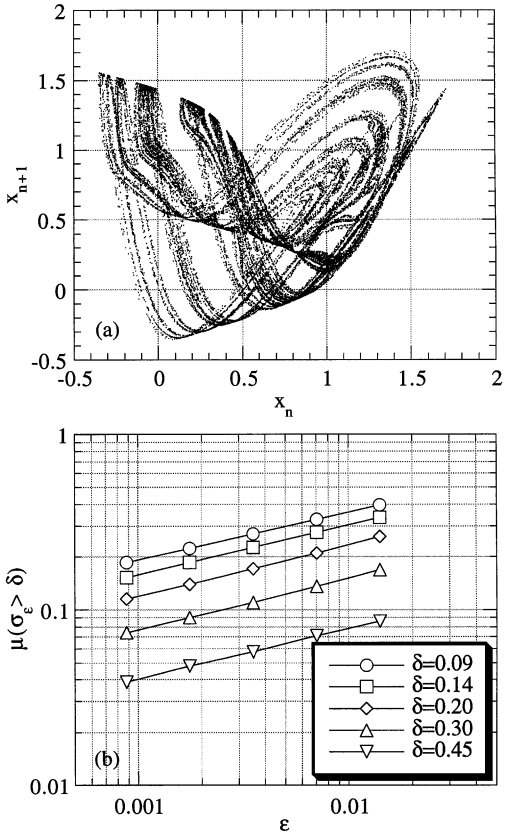

FIG. 3. (a) Ikeda map ($a = 1.0, b = 0.9, \kappa = 0.44, \eta = 6.0$) as observed by two-dimensional time delay of the x coordinate. (b) Scaling of the measure $\mu(\sigma_\varepsilon > \delta)$ of points with prediction error larger than $\delta > 0$. The scaling exponent 0.29 is independent of δ and in good agreement with the exponent $M - D_1 = 2 - 1.71 = 0.29$ given in Conjecture 2. (10^8 time series points were used.)

self-intersections of the attractor as depicted in Fig. 2(a). In Fig. 2(b), the measure $\mu(\sigma_\varepsilon > \delta)$ is depicted as a function of ε for various error bounds δ. $\mu(\sigma_\varepsilon > \delta)$ scales with ε independent of δ, yielding a scaling exponent 0.74 that is in good agreement with the conjectured value $M - D_1 = 2 - 1.26 = 0.74$ (Conjecture 2). D_1 was calculated using both box counting and the Kaplan-Yorke conjecture, and these methods yielded the same result.

Analogous results have been obtained for the Ikeda map [12]

$$z_{n+1} = a + bz_n \exp\left(i\kappa - \frac{i\eta}{1 + |z_n|^2}\right) \qquad (8)$$

($a = 1.0$, $b = 0.9$, $\kappa = 0.44$, and $\eta = 6.0$). With $z_n = x_n + iy_n$ a complex number, the real and imaginary parts of Eq. (8) yield a two-dimensional real map. The measurement space X was constructed as two-dimensional time delays of the x_n component of the state ($M = 2$) as shown in Fig. 3(a). $\mu(\sigma_\varepsilon > \delta)$ is found to scale with exponent 0.29 independent of δ, in good agreement with the exponent $M - D_1 = 2 - 1.71 = 0.29$ predicted by Conjecture 2 [Fig. 3(b)]. As for the Hénon map, both box counting and the Kaplan-Yorke conjecture yielded the same D_1 used here.

Thus the scaling law in Conjecture 2 was verified numerically in both examples. As the diameter ε of the neighborhood used for prediction tends to zero, reliable prediction is found to be possible from most of the points on a self-intersecting image of the attractor in measurement space ($M > D_1$). The fraction of state space measure from which prediction fails vanishes as ε^{M-D_1}.

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[8] As a function of the dimension M of measurement space, the "false nearest neighbor" method determines the data points that are close to each other in measurement space due to self-intersections of the attractor. The fraction of such points can then be determined as a function of the embedding dimension M. Our present work attempts to quantify how this fraction is determined by the attractor's geometry, in particular, by its information dimension.
[9] The minimum required size of the prediction neighborhood scales with the length N of the time series as N^{-1/D_1}. Therefore, a smaller and smaller minimum prediction neighborhood corresponds to longer and longer time series.
[10] The scaling behavior under investigation does not depend on the details of the method. For simplicity, we use the zeroth-order local prediction. The same scaling would also, for example, be obtained using local linear prediction.
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