# Resonance Phenomena of a Solitonlike Extended Object in a Bistable Potential

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We investigate the dynamics of a soliton that behaves as an extended particle. The soliton motion in an effective bistable potential can be chaotic in a similar way as the Duffing oscillator. We generalize the concept of geometrical resonance to spatiotemporal systems and apply it to design a nonfeedback mechanism of chaos control using localized perturbations. We show the existence of *solitonic stochastic resonance.* [S0031-9007(98)05332-0]

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The soliton dynamics in inhomogeneous media [1–5], the spatiotemporal chaos [4,6–10], the chaos control [11–15], and diverse types of resonance phenomena [16–22] have become the object of intensive study in recent years (although rarely have these phenomena been studied simultaneously).

In the present Letter we investigate a solitonic model perturbed by inhomogeneous forces [4,5], for which the exact solution describing the stationary equilibrium solitonic state can be obtained. Furthermore, the stability problem for this solution can be solved exactly. The effective potential for the soliton motion can be bistable depending on the system parameters. Using an additional external periodic force we can have the soliton jumping chaotically between the two potential wells (like in the Duffing oscillator) [23]. We prove that changes of the soliton shape and the waveform of the external perturbations are very relevant to the soliton's dynamics as a whole. We generalize the concept of geometrical resonance [22] to spatiotemporal systems. This concept can be used to design a nonfeedback mechanism of chaos control [13,14]. In our case, this mechanism can be applied in a localized way in space. Finally, we show the existence of solitonic stochastic resonance (SSR).

We are interested in equations of the type

$$\phi_{tt} - \phi_{xx} + R(\phi, \phi_t, x) - G(\phi) = F(x) + P(\phi, \phi_t) q(x,t), \qquad (1)$$

where $G(\phi) = -\frac{\partial U(\phi)}{\partial \phi}$; $U(\phi)$ is a potential that possesses at least two minima [24,25]; $R(\phi, \phi_t, x)$ is a dissipative term, which in general can be nonlinear [9], and the damping coefficient can depend on $x$; $F(x)$ is an inhomogeneous force, and $P(\phi, \phi_t) q(x,t)$ is a general temporal perturbation. Many systems are described by this type of equations, including charge density waves, Josephson junctions, and structural phase transitions [26].

Problems related to Eq. (1) present extraordinary mathematical difficulties. Even if we are to solve [for case $P(\phi, \phi_t) q(x,t) \equiv 0$] the stability problem of a stationary soliton placed on an equilibrium position created by the inhomogeneities [4,5], this task requires the making of big approximations [2]. The most common approximation is to consider the soliton as a structureless pointlike particle [2] (in this case the coordinate of the soliton center of mass and its velocity are the only dynamical variables). Although this is a valid approximation in many cases, our work [4,5] has shown that, in general, the internal dynamics plays a fundamental role.

Besides the applications of the $\phi^4$ equation in phase transition theory [26], it is a *model system* for topological solitons in general. Let us consider as an example the perturbed $\phi^4$ model,

$$\phi_{tt} - \phi_{xx} + \gamma \phi_t - \tfrac{1}{2}(\phi - \phi^3) = F(x), \qquad (2)$$

where

$$F(x) = F_1(x) \equiv \frac{1}{2} A(A^2 - 1) \tanh(Bx) + \frac{1}{2} A(4B^2 - A^2) \frac{\sinh(Bx)}{\cosh^3(Bx)}.$$

The inhomogeneities are chosen in order to have some important properties. The exact stationary solution for the soliton equilibrated by the inhomogeneities in the point $x = 0$ can be obtained $\phi_k = A \tanh(Bx)$. The stability problem of this solution can be solved exactly. The force $F_1(x)$ possesses the interesting property to be topologically equivalent (in the sense of catastrophe theory [27]) to a pitchfork bifurcation, allowing us to have an effective potential (for the soliton) with one or two wells, depending on the system parameters. This system is generic and structurally stable. Therefore the results obtained in this Letter can be generalized qualitatively to other systems topologically equivalent to that described by Eq. (2). The force $F(x)$ permits us to observe important phenomena which occur with the participation of the internal soliton dynamics including the appearance of a great number of internal modes and the soliton destruction due to the instability of the shape mode in some special situations [4,5].

The stability analysis [4,5] which considers small perturbations around the soliton $[\phi(x,t) = \phi_k(x) + f(x) e^{\lambda t}]$ leads to the eigenvalue problem $\hat{L} f = \Gamma f$, where $\hat{L} = -\partial_x^2 + (\frac{3}{2}A^2 - \frac{1}{2} - \frac{3A^2}{2\cosh^2(Bx)})$, $\Gamma = -(\lambda^2 + \gamma \lambda)$.

  

The eigenvalues of the discrete spectrum are given by the expression $\Gamma_n = -\frac{1}{2} + B^2(\Lambda + 2\Lambda n - n^2)$, where $\Lambda(\Lambda + 1) = \frac{3A^2}{2B^2}$. The stability condition of the equilibrium point $x = 0$ (for the soliton) is defined by the eigenvalue of the translational mode $[f_0(x) = \cosh^{-\Lambda}(Bx)]$: $B^2\Lambda - \frac{1}{2} > 0$. A global topological analysis and the investigation of the infinity [5] allow us to have complete qualitative information of the soliton dynamics. When the equilibrium position for the soliton is unstable and the absolute value of the eigenvalue corresponding to the translational mode is sufficiently high, the first shape mode $[f_1(x) = \sinh(Bx)\cosh^{-\Lambda}(Bx)]$, and even other internal modes, can be unstable too, producing the soliton destruction [5]. We have at least two interesting situations: If $A^2 > 1$ and $2\Lambda B^2 > 1$, then there exists only one stable equilibrium point for the soliton ($x_{00} = 0$). If $A^2 > 1$ and $2\Lambda B^2 < 1$, a pitchforklike bifurcation occurs, and we have two stable equilibrium points ($x_{01} < 0$, $x_{02} > 0$) (the point $x_{00} = 0$ becomes unstable). It is important to stress that the bifurcation does not occur at the point we would expect when considering the soliton as a structureless point particle [in this case the number of equilibrium positions depend on the number of zeros of force $F(x)$].

If the system, in the bistable case, is perturbed by an additional periodic force fitted to the shape of the translational mode [4,9] [in Eq. (1) we put $P(\phi, \phi_t) \equiv 1$], then we can have similar situations to those of the Duffing oscillator [23]. But remember that in this case what is jumping between the potential wells is an extended object with a very complicated internal dynamics. On the other hand, for sufficiently high values of $|\Gamma_0|$ for the unstable position, the soliton can bifurcate [5] in an antisoliton (which would feel the middle position as a stable one), and two solitons, each of which would move towards one of the wells.

The recently formulated concept of geometrical resonance (GR) [22] can be very useful in this context. Generalizing this concept for spatiotemporal systems we define $\phi_{\mathrm{GR}}(x,t)$ as a GR solution of Eq. (1) if it satisfies the condition

$$R(\phi_{\mathrm{GR}}, \partial_t\phi_{\mathrm{GR}}, x) = P(\phi_{\mathrm{GR}}, \partial_t\phi_{\mathrm{GR}})q_{\mathrm{GR}}(x,t). \quad (3)$$

In this case there exists (local) conservation of energy,

$$H \equiv \int_{-\infty}^{\infty}\left[\frac{1}{2}\left(\frac{\partial\phi_{\mathrm{GR}}}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi_{\mathrm{GR}}}{\partial x}\right)^2 + U(\phi_{\mathrm{GR}}) - F(x)\phi_{\mathrm{GR}} + C\right]dx. \quad (4)$$

If we want to observe a GR situation, we should use a $q_{\mathrm{GR}}(x,t)$ such that it satisfies the GR condition (3).

Consider the Eq. (1) with $R = \gamma\phi_t$, $P = 1$, $G(\phi) = \frac{1}{2}(\phi - \phi^3)$, and $F(x)$ as defined in (2).

We assume that the stability condition $B^2\Lambda - \frac{1}{2} > 0$ for the equilibrium position $x = 0$ holds. In this case we can write an approximate solution for $\phi(x,t)$ ($q \equiv 0$),

$$\phi(x,t) = A\tanh(Bx) + \frac{h_{00}\cos(\omega_0 t + \theta_0)}{\cosh^{\Lambda}(Bx)}, \quad (5)$$

where $\omega_0^2 = \Gamma_0$.

Thus, for small values of $h_{00}$ the perturbation

$$q_{\mathrm{GR}}(x,t) = -\frac{\omega_0 h_{00}\gamma\sin(\omega_0 t + \theta_0)}{\cosh^{\Lambda}(Bx)} \quad (6)$$

satisfies the GR condition. This explains a whole series of experiments performed in [5,9]. There the $\phi^4$ kink confined in a potential well created by the inhomogeneity $F(x)$ was perturbed by the time-dependent force (6). The authors have got resonances of the translational mode practically without deformation of the kink profile. Chaotic behavior is obtained only for high values of the perturbation amplitude, when (5) is not a GR solution anymore. Meanwhile, if we use a different time-dependent force [e.g., $q(x,t) = \rho_0\cos\omega t$] which does not satisfy the GR condition, we will obtain irregular behavior in time and space with much smaller amplitudes.

The GR [22] provides a mechanism for nonfeedback control of chaos.

Suppose we have the following equation:

$$\phi_{tt} - \phi_{xx} + \gamma\phi_t - \frac{1}{2}(\phi - \phi^3) = F(x) - P_0\frac{\cos(\omega t)}{\cosh^2(Bx)} + F_c(x,t). \quad (7)$$

We assume that for certain initial conditions Eq. (7) is in a chaotic regime provided $F_c(x,t) \equiv 0$.

We are left with the problem of eliminating the chaotic motion using control $F_c(x,t)$. Additionally, it is expected that the control driving term is small and localized in space. In order to do this we should select the control term in such a way that the solution will be sufficiently close to a $T'$ periodic GR solution.

We can choose the control function in the form

$$F_c(x,t) = \frac{g_c\cos(\omega_c t + \theta_c)}{\cosh^{\Lambda}(Bx)}. \quad (8)$$

Using the concept of "almost adiabatic invariant" [22,28], we arrive at a condition for the control parameters

$$\left\langle\int_{-\infty}^{\infty} -\gamma\left(\frac{\partial\phi_{\mathrm{GR}}}{\partial t}\right)^2 - P_0\frac{\partial\phi_{\mathrm{GR}}}{\partial t}\frac{\cos(\omega t)}{\cosh^2(Bx)} + g_c\frac{\partial\phi_{\mathrm{GR}}}{\partial t}\frac{\cos(\omega_c t + \theta_c)}{\cosh^{\Lambda}(Bx)}dx\right\rangle_{T'} = 0. \quad (9)$$

The fact that we have taken (8) associated with the translational mode (which approximately satisfies the GR condition) always allows us to find control parameters (with small $g_c$) in order to suppress chaos. An Arnold-like tongue structure, similar to that observed in [22], can be obtained here. In Fig. 1 we show the results of the application of the localized nonfeedback control.

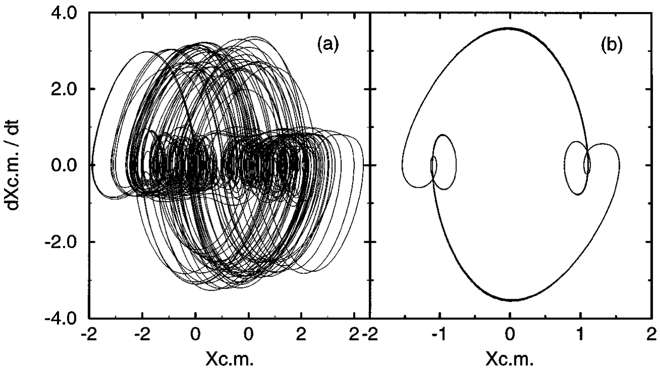


FIG. 1. Phase portraits for the motion of the soliton center of mass in Eq. (7); $F(x)$ is defined as in Eq. (2) ($B = 1/2$). (a) Chaotic motion in the absence of control [$F_c(x,t) \equiv 0$], $\gamma = 0.1$, $A = 1.5$, $P_0 = 1$, and $\omega = 0.65$. (b) Periodic motion as a result of the application of nonfeedback control, $\omega_c = 0.065$, $g_c = 0.35$, and $\theta_c = 0$.

Note that the validity of the concept of GR as a mechanism for chaos control is not limited to the case in which the unperturbed equation can be solved exactly. Using some topological analysis it is possible to guess the "shape" of the GR solution, and then through the concept of almost adiabatic invariant ($\langle \frac{dH[\phi_{\mathrm{GR}}]}{dt} \rangle_{T'} \approx 0$) one can obtain the conditions such that the solution will be close to a GR solution inside of a "mode-locked" tongue.

The system we have presented in this Letter with a solitonlike extended object moving in an effective bistable potential can also be useful for studying other phenomena, e.g., the spatiotemporal stochastic resonance (SR) for the motion of an extended state with complicated internal dynamics.

Consider the equation

$$\phi_{tt} - \phi_{xx} + \gamma\phi_t - \tfrac{1}{2}(\phi - \phi^3) = F(x) + q(x,t), \tag{10}$$

where

$$F(x) = \begin{cases} B(4B^2 - 1)\tanh[B(x + x_0)] & \text{for } x < 0, \\ 0 & \text{for } x = 0, \\ B(4B^2 - 1)\tanh[B(x - x_0)] & \text{for } x > 0; \end{cases} \tag{11}$$

$$q(x,t) = \begin{cases} [P_0\cos(\omega t) + \eta(x,t)]\cosh^{-2}[B(x + x_0)] & \text{for } x < 0, \\ [P_0\cos(\omega t) + \eta(x,t)]\cosh^{-2}[B(x - x_0)] & \text{for } x > 0; \end{cases} \tag{12}$$

here $4B^2 > 1$. The forces $F(x)$ and $q(x,t)$ are defined by Eqs. (11) and (12) in such a way that there are two equilibrium points for the soliton and the motion in each well is very close to the GR condition. In the absence of the white noise $\eta(x,t)$ [$\langle \eta(x,t) \rangle = 0$, $\langle \eta(x,t)\eta(x',t') \rangle = 2D\delta(x - x')\delta(t - t')$], the periodic force above is unable to make the soliton jump between the wells.

When we switch on the noise, it is possible to observe a maximum in the graph of the signal-to-noise ratio (SNR) versus $D$ (see Fig. 2). Here as the "signal" we take the time series of the soliton center-of-mass coordinate: $x_{\mathrm{c.m.}} = \frac{\int_{-l/2}^{l/2} x\phi_x^2\,dx}{\int_{-l/2}^{l/2} \phi_x^2\,dx}$. The SNR is measured following the traditional method [18].

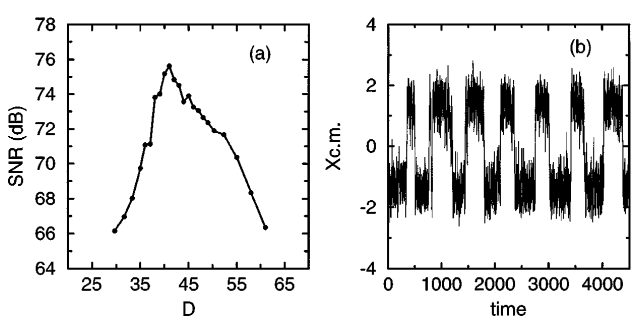


FIG. 2. Two aspects of the solitonic stochastic resonance: (a) There is a maximum of the SNR at a critical value of noise intensity. (b) Synchronization of the stochastic jumps with the periodic perturbation linked with translational mode of the soliton. ($P_0 = 0.04$, $\omega = 0.01$, $x_0 = 2.5$, and $B = 0.7$.)

The maximum synchronization is obtained with a signal associated with the translational mode. In this context, Fig. 3 shows that the deformation of the kink profile at the SSR is minimal. These results allow us to predict the optimum entertainment of the noise by means of a periodic signal, not necessarily a simple sinusoidal signal but a more complex spatiotemporal function. The stochastic resonance will depend on the characteristic shape of the kink and the waveform of the perturbation (in time and space). A manifestation of this fact is the following phenomenon: if instead of the translational mode in perturbation (12) we use the first shape mode, then we do not observe SSR.

Even when $F(x)$ has three zeros, if we are in the presence of an extended object, the bistability is not a sufficient condition for the stochastic resonance. The extended object should "feel" the bistability, and the internal modes should be stable in the vicinity of all the equilibrium points, including the central equilibrium point which is unstable for the translational mode. On the other hand, if the eigenvalue $|\Gamma_0|$ corresponding to the unstable equilibrium position in the bistable potential is very high, then the first shape mode can be unstable too and the soliton will bifurcate in two solitons and one antisoliton. This is what occurs when the parameters in Eq. (11) are such that $B^2(4B^2 - 1)\tanh(Bx_0) > \frac{23}{50}$. The center of mass of this "three-particle structure" is always oscillating around $x = 0$, and therefore there is no SSR.

The spatiotemporal stochastic resonance has been considered in some recent papers [19,21]. In particular,

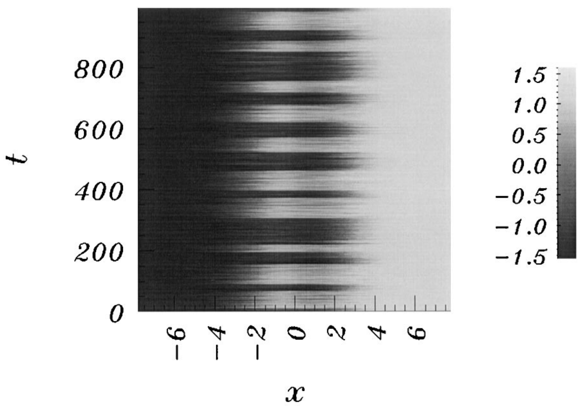


FIG. 3. Spatiotemporal evolution of the kink soliton at the stochastic resonance. Note that there is a minimal deformation of the kink profile.

Ref. [19] deals with the $\phi^4$ equation. However, the SSR presented here is a completely new phenomenon: there the behavior of the field $\phi$ is taken for the signal and the important bistability is in the potential $U(\phi)$, whereas in our work the relevant signal is the time series for the coordinate of the soliton's center of mass. In Ref. [19] there is no inhomogeneous force $F(x)$. The spatiotemporal stochastic resonance studied there does not depend on $F(x)$. In our case the potential $U(\phi)$ is important for the existence of the soliton solution [24], but it is the bistability in $x$, created by $F(x)$ which is the key for the SSR. And, as we have seen in this Letter, not for every $F(x)$ the SSR exists.

In general, we prove that changes of the soliton shape are very relevant to its dynamics as a whole. Additionally, the waveform of the perturbation is crucial for all the resonance phenomena, including the nonfeedback mechanism of the chaos control and the stochastic resonance. We believe that these phenomena are relevant to other systems where there is an extended state with a complicated internal dynamics moving in a nonlinear potential force.

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