Gravitational Lensing of Gravitational Waves from Inspiraling Binaries by a Point Mass Lens

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Gravitational lensing of gravitational waves from inspiraling binaries is discussed in the context of advanced laser-interferometer detectors, taking correct account of the diffraction effect. Convolving the spectrum of the lensed waveform with that of the detector noise, we calculate how much the signal-to-noise ratio is magnified by gravitational lensing as a function of the mass and position of the lens. When the lens is much lighter than $\sim 10^2 M_\odot$, the diffraction is so effective that the wave flux is not magnified appreciably. We predict that lensed waveforms are distinguishable from unlensed ones in that the signal-to-noise ratio shows an oscillatory behavior as the frequency sweeps up. [S0031-9007(97)05167-3]

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The gravitational waves emitted during the inspiral stage before the final coalescence of two neutron stars (NS-NS binary) is the most promising source for the laser-interferometer detectors under construction such as the LIGO [1]. To this end, much effort has been made to calculate accurately the waveform template [2] which is matched filtered with the observed signal, and also to estimate the event rate [3]. Thus it is important to consider every possible source of noise that alters the results of these theoretical calculations. One such kind of noise may be the gravitational focusing of gravitational waves [4], the subject we shall investigate in this paper.

Previously, Wang *et al.* [5] considered the microlensing of gravitational waves from inspiraling binaries by (hypothetical) stellar mass lenses distributed over the Universe (since the probability of lensing by galaxies is negligible), and concluded that, due to the lensing magnification, there may be a significant number of those inspiral events which would be too distant to be detected had it not been for the lensing. They simply calculated the lensing magnification using geometric optics limit in the same way as for the optical light, and took the maximum magnification to be infinite on the caustics. As shown by several authors [6], however, if the wavelength λ is longer than the Schwarzschild radius of the lens mass M, then the wave effect (diffraction) becomes important so the maximum magnification should be small. Since the wavelength is typically $\sim 10^2 - 10^4$ km in the observable frequency range of the planned detectors, the wave flux cannot be magnified significantly for lens masses lighter than $\sim 10^2 M_\odot$, as is the case for the microlensing considered in Ref. [5].

This paper provides the first proper treatment of gravitational lensing of gravitational waves from inspiraling binaries in the context of advanced laser-interferometer detectors, taking correct account of the diffraction effect. We show that the lensed events should be much rarer than estimated in Ref. [5] for the LIGO type detectors, but that it is possible to distinguish lensed waveforms from unlensed ones once the former are actually detected. We assume the $\Omega = 1$ and $\Lambda = 0$ cosmology and use the units $c = G = 1$.

To give a rough account of why the ratio M/λ determines the significance of diffraction, consider monochromatic waves with wavelength λ passing through a double slit with the slit width of order the Einstein radius $\xi_{\rm E} \sim (MD)^{1/2}$ where D is the distance from us to the lens. Interference of two waves coming from the slits would produce an oscillating pattern in the intensity distribution on the screen located at the observer. The width ℓ of the central peak of the distribution is given by $\ell \sim \lambda D/\xi_{\rm E} \sim (D/M)^{1/2}\lambda$. Then the maximum magnification of the wave flux is on the order of $\sim \xi_{\rm E}/\ell \sim M/\lambda$. Thus large magnification can occur only when the wavelength is much shorter than the Schwarzschild radius of the lens.

More quantitatively, we perform the phase integral according to the Huygens principle in the wave optics theory to obtain the wave amplitude at the observer. Consider a point source of radiation whose position vector on the source plane is $\vec{\eta}$, and a Schwarzschild lens of mass M located at redshift of $z_{\rm L}$. The arrival time at the observer for a ray passing through $\vec{\xi}$ on the lens plane is written as [7]

$$\tau(\vec{x},\vec{y}) = 4M(1+z_{\rm L})\left[\tfrac{1}{2}|\vec{x}-\vec{y}|^2 - \ln x - \phi_{\rm m}(y)\right], \tag{1}$$

where $\vec{x} := \vec{\xi}/\xi_{\rm E}$ and $\vec{y} := \vec{\eta}/\eta_{\rm E}$ are dimensionless variables normalized by the Einstein radius, and the origins of $\vec{x}$ and $\vec{y}$ planes are on the optic axis which connects the observer and lens. The first and second terms of Eq. (1) represent the geometrical and gravitational time delays, respectively, and we choose $\phi_{\rm m}(y)$ so that the minimum value of τ on the $\vec{x}$ plane is zero. For a Fourier component of waves with the frequency f at the observer, the wave amplitude (apart from the time-dependent $e^{-2\pi i f t}$ factor) is given by the Kirchhoff integral on the lens plane as [7]

$$\psi(\hat{\omega}, y) = \frac{\hat{\omega}}{i\pi}\int d^2x \, \exp[2\pi i f \tau(\vec{x},\vec{y})] \tag{2}$$

$$= \exp\{\tfrac{1}{2}\pi\hat{\omega} + i\hat{\omega}[\ln\hat{\omega} - \phi_{\rm m}(y)]\} \times \Gamma(1 - i\hat{\omega}) F(i\hat{\omega}, 1; i\hat{\omega}y^2), \tag{3}$$

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where $\hat{\omega} := 4\pi f \hat{M}$ measures the ratio of the Schwarzschild radius of the lens to the wavelength, $\hat{M} := M(1 + z_L)$ is the redshifted lens mass, and F is the confluent hypergeometric function. The prefactor of Eq. (2) is chosen such that $\psi = 1$ without lensing. Then the lensing magnification $\tilde{\mu} = |\psi|^2$ for the Fourier component is

$$\tilde{\mu}(\hat{\omega}, y) = \tilde{\mu}_m(\hat{\omega}) \, |F(i\hat{\omega}, 1; i\hat{\omega} y^2)|^2, \tag{4}$$

where $\tilde{\mu}_m(\hat{\omega}) := 2\pi\hat{\omega}/(1 - e^{-2\pi\hat{\omega}})$ is the maximum magnification which occurs when the source is just behind the lens $y = 0$. Thus one again verifies the fact that $\tilde{\mu}_m \gg 1$ only when $\hat{\omega} \gg 1$ as shown above. It is interesting to note that Eq. (3) can also be derived by solving the propagation equation $\Box\psi = 0$ with the metric of linearized gravity, which yields the Schrödinger equation in a Coulomb scattering potential [8].

The assumption of a point source is highly justified for radiation from a binary so it is unnecessary to average Eq. (3) over y, since the source size is smaller than the wavelength because of its quadrupolar character. In fact, the change in the time delay $\tau(\vec{x}, \vec{y})$ due to the source size $\eta_* \, (<\lambda)$ is $\sim \tau\eta_*/\eta_E \sim (M/D)^{1/2}\eta_* \ll f^{-1}$, implying that the phase in Eq. (2) does not move significantly over the source size. Also it is evident that, even when the lens is not a point mass but has finite size, the above formulas are valid if the lens size is much smaller than the Einstein radius ξ_E (typically greater than 100 AU for cosmological lensing).

Before examining the lensing effect on gravitational waves, let us briefly summarize the notation of some quantities which characterize the laser-interferometer detectors. Performing the matched filtering between the incoming waveform $h(t)$ to the detector and the trial waveform $h_*(t)$ in templates, the signal-to-noise ratio (SN) is given by $\rho[h, h_*] = \langle h|h_*\rangle/\langle h_*|h_*\rangle^{1/2}$ [9], where the inner product is defined as

$$\begin{aligned} \langle a|b\rangle &:= 4\int_0^\infty df \, \frac{\mathrm{Re}[\tilde{a}^*(f)\tilde{b}(f)]}{S_n(f)}, \\ \tilde{a}(f) &:= \int_{-\infty}^{\infty} dt \, a(t) e^{2\pi i f t}, \end{aligned} \tag{5}$$

and $S_n(f)$ is the power spectrum of noise in the detector. For the advanced LIGO detector, we use [10]

$$f S_n(f) = Ax/g(x), \qquad g(x) := \frac{x^4 \mathrm{H}(x - x_{\min})}{1 + 2x^4(1 + x^2)}, \tag{6}$$

where $x := f/f_k$, $f_k = 70$ Hz, $A = 2.1 \times 10^{-46}$, $x_{\min} = 1/7$, and $\mathrm{H}(x)$ is the step function. The maximum SN is achieved when the trial waveform coincides with the incoming waveform: $\rho_m[h] = \rho[h, h] = 2[\int_0^\infty df |\tilde{h}(f)|^2/S_n(f)]^{1/2}$. For waves from a binary at redshift of z in circular orbit, the lowest order (Newtonian) quadrupole formula gives the expression for the maximum SN as [9,10]

$$\rho_m[h] = \frac{(5/6)^{1/2}}{4\pi^{2/3}} \frac{\Theta \hat{\mathcal{M}}}{d_L(z)} \left[\int_0^{f_m} \frac{df}{f} \frac{(\hat{\mathcal{M}} f)^{-1/3}}{f S_n(f)} \right]^{1/2}, \tag{7}$$

where $d_L(z) = 2cH_0^{-1}(1 + z)[1 - (1 + z)^{-1/2}]$ is the luminosity distance from us to the source redshift z, $\hat{\mathcal{M}} = \mathcal{M}(1 + z)$ is the redshifted chirp mass ($\mathcal{M} = [(M_1M_2)^3/(M_1 + M_2)]^{1/5}$ for the binary masses M_1 and M_2), Θ is the orientation function depending on the angular position and inclination of the source ($0 \le \Theta \le 4$), and f_m is the maximum frequency at the innermost stable circular orbit. The post-Newtonian calculation of point mass limit shows that $f_m = (1.42 \text{ kHz})(1.4M_\odot/M_1)/(1 + z)$ for equal-mass binaries $M_1 = M_2$ [11].

When gravitational waves are lensed by an intervening point mass lens, each Fourier component of the observed (lensed) waveform $\tilde{h}_L$ is the product of the unlensed waveform $\tilde{h}$ and the wave function ψ given in Eq. (3); $\tilde{h}_L = \tilde{h}\psi$. Consequently the maximum SN^2 is magnified due to gravitational lensing by the factor [cf. Eq. (4)]

$$\mu(m, y) := \frac{\rho_m^2[h_L]}{\rho_m^2[h]} = \frac{\int_0^{x_m} dx \, x^{-7/3} g(x) \tilde{\mu}(mx, y)}{\int_0^{x_m} dx \, x^{-7/3} g(x)}, \tag{8}$$

where $m := 4\pi f_k \hat{M}$. This is plotted in Figs. 1(a) and 1(b) versus the source position y and the redshifted lens mass $\hat{M}$. Also plotted is the magnification factor in geometric optics limit, $\mu_g(y) = (y^2 + 2)/[y(y^2 + 4)^{1/2}]$ [7], which is independent of the lens mass. The results are very insensitive to the upper limit of the integrals $x_m := f_m/f_k$ when $x_m \gtrsim 10$, and in all the figures we set $x_m = 15$ (corresponding to a binary of $M_1 \sim M_2 \sim 1.4M_\odot$ at redshift $z \sim 0.35$ [11]). The figure shows the tendency that geometric optics becomes good as the lens mass increases and would be valid for lensing by galactic masses. In case of microlensing by stellar masses, however, the diffraction is so effective that the lensing cross section ($\propto y^2$) for large magnification is negligibly small. Therefore we conclude that the number of high-redshift, highly magnified inspiral events detectable by the advanced LIGO is overestimated in Ref. [5], which uses geometric optics for the magnification and considers microlensing for the lensing probability.

The most important information in the matched filtering is the phase of the wave signal [10]. Suppose that we detect a lensed signal and we do not have lensed waveforms in templates. Then we cannot get the possible maximum value of SN since the phase difference between the lensed and unlensed waveforms [i.e., the phase of ψ in Eq. (3)] changes with time as the frequency sweeps up. If our templates include unlensed waveforms only, SN would decrease typically by the following factor:

$$\frac{\rho[h_L, h]}{\rho_m[h_L]} = \frac{\int_0^{x_m} dx \, x^{-7/3} g(x) \mathrm{Re}[\psi(mx, y)]}{\mu^{1/2} \int_0^{x_m} dx \, x^{-7/3} g(x)}, \tag{9}$$

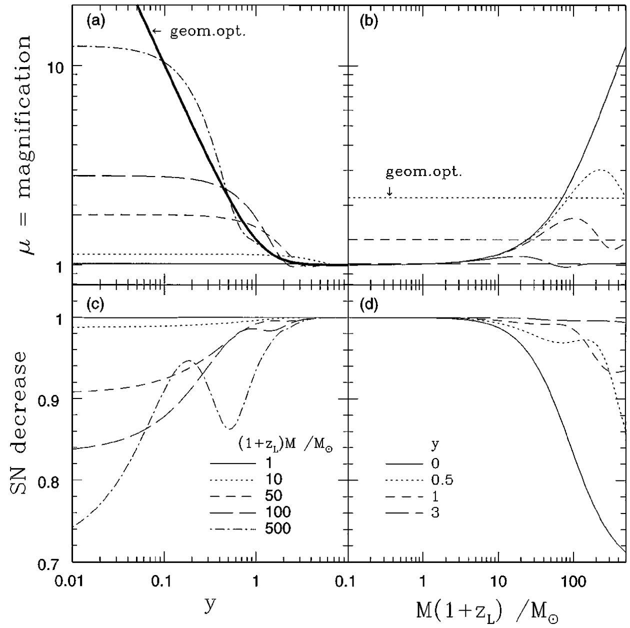

FIG. 1. (a),(b) The lensing magnification factor of the signal-to-noise ratio squared [Eq. (8)]. (c),(d) The decreasing factor in the signal-to-noise ratio [Eq. (9)] when the lensed waveform h_{L} is matched filtered with the unlensed waveform h. The abscissae in (a) and (c) are the source position y in units of the Einstein radius and in (b) and (d) the redshifted lens mass $\hat{M} = M(1 + z_{\mathrm{L}})$. The magnification factor in the geometric optics limit is plotted by the thick curve in (a) and by horizontal lines in (b).

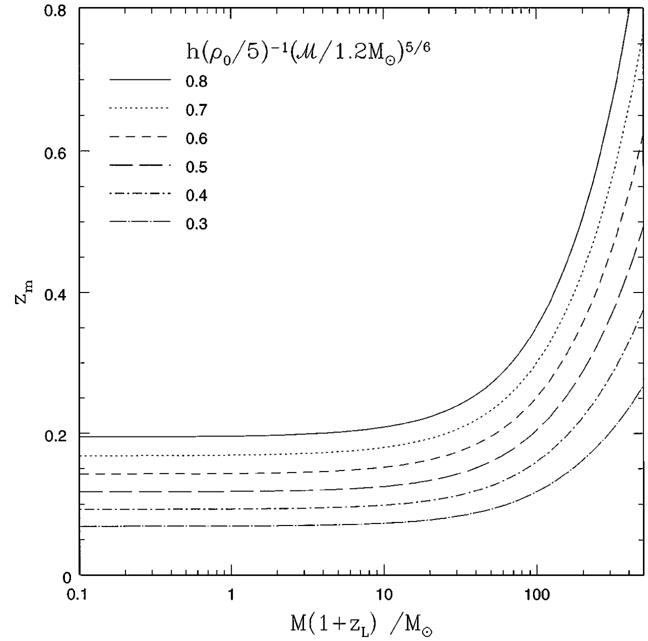

FIG. 2. The maximum redshift reached by the advanced LIGO detector taking the lensing magnification into account [Eq. (10)] versus the redshifted lens mass $\hat{M}$ for some values of $h(\rho_0/5)^{-1}(\mathcal{M}/1.2M_\odot)^{5/6}$. Equation (6) was used for the noise spectrum.

which is plotted in Figs. 1(c) and 1(d). Note that in geometric optics the phase shift due to lensing remains constant and so SN does not decrease. Certainly it is desirable to be equipped with the lensed waveform in our templates to achieve the maximum SN for every signal. Moreover it may be possible to determine the lens mass from the matched filtering of the lensed signal with such templates if this SN decrease is sufficiently large. However, the figure indicates that the SN decrease due to diffraction is insignificant except for some rare cases of close encounter of sources with very large lens mass, occurring with extremely low probability (discussion on the probability is given below).

Next we calculate the maximum redshift z_{m} of sources reached by the single advanced LIGO detector with SN greater than a certain threshold ρ_0, in the presence of the lensing magnification. Solving $\rho_{\mathrm{m}}[h_{\mathrm{L}}] = \rho_0$ with $\Theta = 4$ and $y = 0$ (maximum magnification) with respect to z yields [12]

$$z_{\mathrm{m}} = (\alpha + \beta + \alpha^2/\beta)^6 - 1\,, \tag{10}$$

where

$$\alpha := \frac{(5/6)^{1/2}}{6\pi^{2/3}} \frac{H_0 \mathcal{M}}{\rho_0} \left[\int_0^{f_{\mathrm{m}}} \frac{df}{f} \frac{(\mathcal{M}f)^{-1/3} \tilde{\mu}_{\mathrm{m}}(\hat{\omega})}{f S_{\mathrm{n}}(f)} \right]^{1/2} \tag{11}$$

and $\beta := [\alpha^3 + \frac{1}{2} + (\alpha^3 + \frac{1}{4})^{1/2}]^{1/3}$. Figure 2 plots the maximum redshift z_{m} versus the redshifted lens mass $\hat{M}$ for various values of $h(\rho_0/5)^{-1}(\mathcal{M}/1.2M_\odot)^{5/6}$ where $h := H_0/(100\ \mathrm{km\,s^{-1}\,Mpc^{-1}})$. Note that $\hat{M} \to 0$ and $\hat{M} \to \infty$ correspond to no lensing limit and lensing in geometric optics limit, respectively. The figure shows that the maximum detectable redshift hardly increases due to microlensing by stellar mass lenses. Thus we conclude that the high-redshift tail in the redshift distribution of inspiral events due to the lensing magnification (Fig. 3 of Ref. [5]) should not exist (but much lower tail due to galaxy lensing may exist).

Finally, we estimate the lensing probability and the lensed event rate very roughly. Following Ref. [5] we consider hypothetical point mass lenses distributed over the Universe, since the probability of galaxy lensing ($\propto z^3$) is an order of magnitude smaller. The optical depth of lensing for source redshift z with impact parameter smaller than y is $\tau = \frac{1}{4}\Omega_{\mathrm{L}} y^2 z^2$ for small z [13], where Ω_{L} is the density parameter contributed from lensing objects. It is estimated in Ref. [5] that unlensed events with SN > 5 have the rate $\dot{N} \sim 150$–$200\ \mathrm{yr}^{-1}$ for $h \sim 0.5$–0.8 and the average redshift $\bar{z} \sim 0.2$. So the lensed event rate is of order $\dot{N}_{\mathrm{L}} \sim \tau\dot{N} \sim (0.16$–$0.2) y^2 (\Omega_{\mathrm{L}}/0.1)\,(z/0.2)^2\ \mathrm{yr}^{-1}$. Thus we expect that a lensed event with $y = 1$ occurs about once in every 5–8 yr if $\Omega_{\mathrm{L}} = 0.1$. These crude arguments may slightly underestimate the number of small y events since we have neglected entirely the increase of events due to the lensing magnification. More accurate calculation of the lensing probability requires detailed information on the mass function of lenses and is not presented here.

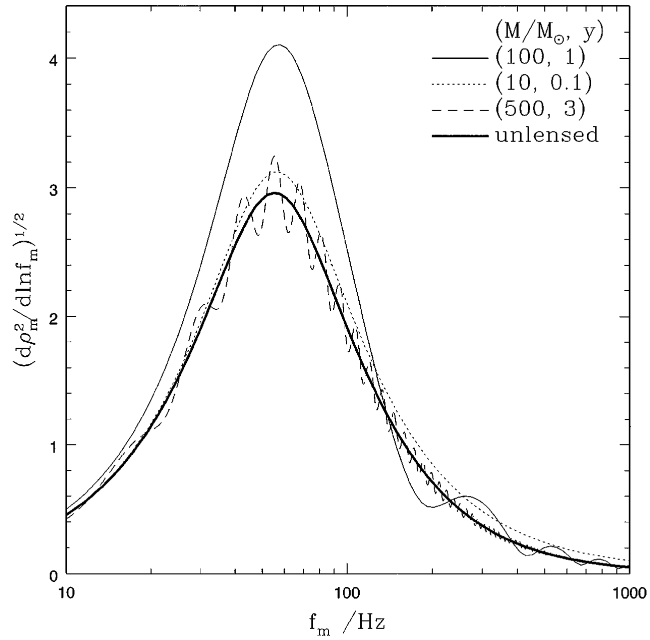

FIG. 3. Frequency distribution of the signal-to-noise ratio for some values of the source position y in units of the Einstein radius and the redshifted lens mass $\hat{M}$, with $h = 0.7$, $\Theta = 1.6$, $\mathcal{M} = 1.2 M_\odot$, and $z = 0.1$. Shown in thick curve is the unlensed "universal" distribution.

Although our results are negative in that the event rate would not significantly increase even with the lensing magnification, in closing let us predict a possible observable effect from diffraction. For unlensed waveforms in the Newtonian formula [Eq. (7)], plots of the maximum SN $\rho_{\rm m}$ versus the sweeping-up frequency $f_{\rm m}$ have universal shape irrespective of individual binaries. In fact, this $f_{\rm m}$-$\rho_{\rm m}$ relation is observable if we filter the observed signal with the function $w(t) = 2 f_{\rm m} j_0(2\pi f_{\rm m} t)$, though the error bar is likely to be very large. On the other hand, lensed waveforms should yield deviation from the universal curve because the magnification factor [Eq. (3)] depends on the frequency. We plot in Fig. 3 the frequency distribution of SN, $(d\rho_{\rm m}^2[h_{\rm L}]/d\ln f_{\rm m})^{1/2}$, versus $f_{\rm m}$ for some values of the source position y and the lens mass $\hat{M}$. Detection of large deviations from the unlensed universal curve, in particular, the oscillatory behavior like those in Fig. 3, is suspected as a signature of gravitational lensing. Though the frequency of the detection of such events is too low to discuss any statistical properties of lensing objects, a single discovery of one such phenomenon—diffraction of gravitational waves—itself is physically very interesting.

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[1] A. Abramovici *et al.*, Science **256**, 325 (1992); for a review, K. S. Thorne, in *300 Years of Gravitation*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1987), p. 330.
[2] For example, C. W. Lincoln and C. M. Will, Phys. Rev. D **42**, 1123 (1990).
[3] E. S. Phinny, Astrophys. J. **380**, L17 (1991); R. Narayan, T. Piran, and A. Shemi, Astrophys. J. **379**, L17 (1991).
[4] D. Marković, Phys. Rev. D **48**, 4738 (1993).
[5] Y. Wang, A. Stebbins, and E. L. Turner, Phys. Rev. Lett. **77**, 2875 (1996).
[6] H. C. Ohanian, Int. J. Theor. Phys. **9**, 425 (1974); P. V. Bliokh and A. A. Minakov, Astrophys. Space Sci. **34**, L7 (1975); R. J. Bontz and M. P. Haugan, *ibid.* **78**, 199 (1981); K. S. Thorne, in *Gravitational Radiation*, edited by N. Deruelle and T. Piran (North-Holland, Amsterdam, 1983), p. 28.
[7] P. Schneider, J. Ehlers, and E. E. Falco, *Gravitational Lenses* (Springer-Verlag, Berlin, 1992), pp. 220 and 240; y^2 in Eqs. (7.10) and (7.11) of this reference should be replaced by $-y^2$.
[8] S. Deguchi and W. D. Watson, Phys. Rev. D **34**, 1708 (1986); Astrophys. J. **307**, 30 (1986); P. C. Peters, Phys. Rev. D **9**, 2207 (1974).
[9] L. S. Finn and D. F. Chernoff, Phys. Rev. D **47**, 2198 (1993); for a good review, E. Poisson, in *Proceedings of the 6th Canadian Conference on General Relativity and Relativistic Astrophysics* (Report No. gr-qc/9508017).
[10] C. Cutler and É. E. Flanagan, Phys. Rev. D **49**, 2658 (1994).
[11] L. E. Kidder, C. M. Will, and A. G. Wiseman, Phys. Rev. D **47**, 3281 (1993).
[12] L. S. Finn, Phys. Rev. D **53**, 2878 (1996); our α [Eq. (11)] corresponds to $2\alpha_0/3$ in Eq. (4.10b) of this reference, but our calculation shows that α_0 in Eqs. (4.10a) and (4.10d) should be replaced by $\alpha_0/8$.
[13] E. L. Turner, J. P. Ostriker, and J. R. Gott III, Astrophys. J. **284**, 1 (1984).