Scarred and Chaotic Field Distributions in a Three-Dimensional Sinai-Microwave Resonator

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For about 200 eigenfrequencies of a Sinai-shaped three-dimensional microwave resonator electromagnetic field distributions were mapped by measuring the eigenfrequency shift $\Delta\nu$ as a function of the position of a spherical perturbing bead. Both regular and chaotic field patterns were found. For the chaotic field distributions all components of the electromagnetic field were found to be uncorrelated and Gaussian distributed. [S0031-9007(97)04780-7]

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Chaotic billiards are frequently used models to study the quantum mechanical properties of classically chaotic systems. There exists a reasonable large number of calculations (see, e.g., Refs. [1–4]) as well as microwave analog experiments [5–7]. In all cases the studied spectral properties, in particular level spacing distribution and spectral rigidity, were in accordance with the predictions of random matrix theory as was conjectured by Bohigas, Giannoni, and Schmit in a ground breaking paper [8]. Exceptions were only parts of the spectra where eigenvalues associated with bouncing-ball orbits disturbed the universal behavior [4,7]. The bouncing balls can be taken into account with the help of the periodic orbit theory where they give rise to additional contributions to the Gutzwiller trace formula [4].

All calculations and experiments mentioned above have been performed in planar billiards. In three-dimensional systems the available information is still scarce. On the theoretical side we are aware of only one quantum mechanical calculation by Primack and Smilansky in a three-dimensional symmetry-reduced Sinai billiard [9]. Their main object was the analysis of the different bouncing-ball contributions to the spectra, in particular with respect to the qualitative differences between two and three dimensions. Frank and Eckhardt calculated the eigenfrequencies of three-dimensional integrable microwave cavities, also with the main emphasis on periodic orbit theory [10]. In accordance with a prediction by Balian and Duplantier [11] they found that because of the two possible polarizations of the electromagnetic waves only orbits with an even number of reflections contribute to the spectra.

On the experimental side there are the acoustic resonance experiments by Ellegaard *et al.* [12] in quartz blocks shaped as three-dimensional Sinai billiards and the microwave experiments by Deus *et al.* [13] and Alt *et al.* [14] in three-dimensional chaotic cavities. In all cases the predictions of random matrix theory were verified though in these experiments, in contrast to two-dimensional microwave resonators, the correspondence to quantum mechanics is no longer given.

The fact that random matrix theory can be applied also to the spectra of three-dimensional microwave resonators is by no means trivial. After all the Schrödinger equation is an equation for the wave functions, whereas in the Maxwell equations the six field components E_x, E_y, E_z, B_x, B_y, and B_z are coupled in a complicated way. It is therefore desirable to have information not only on the spectra but also on the field distributions, and to see whether the phenomenology known from scalar fields is found here, too.

To measure such field distributions we applied the perturbing bead method used already to map wave functions in two-dimensional cavities [6] to an octant of a three-dimensional Sinai billiard, built as a copper box of inside dimensions $l_x = 96$ mm, $l_y = 82$ mm, $l_z = 106$ mm with an octant of a sphere of radius $r = 39$ mm inserted (see Fig. 1). The microwaves were fed into the resonator through an antenna positioned in the center of a wall (the right one in Fig. 1). A metallic sphere of radius $R = 5.5$ mm, supported by a thread, could be moved within the volume by changing the length of the thread and shifting the top plate. By varying the position of the bead on a three-dimensional rectangular array in steps of 12, 10, and 2.4 mm in the x, y, and z directions, respectively, about 2000 different spectra were taken, each of them containing more than 200 eigenfrequencies.

In the measurement the fact was used that the presence of the bead gives rise to a shift $\Delta\nu$ of the eigenfrequencies from which information on the fields at the position of the bead can be obtained. Though the technique has been known for a long time [15] the essentials shall be sketched here to make the Letter self-explanatory. Let $\mathbf{E}_{n_0}$ and $\mathbf{E}_n$ be the electric fields for an eigenresonance without and in the presence of the bead. Both fields obey Helmholtz equations

$$(\Delta + k_{n_0}^2)\mathbf{E}_{n_0} = 0, \qquad (\Delta + k_n^2)\mathbf{E}_n = 0, \tag{1}$$

which have to be solved with the boundary condition $(\mathbf{E}_{n_0})_\parallel = (\mathbf{E}_n)_\parallel = 0$ on the surface of the cavity. Combination of the two equations and application of Green's theorem in the usual way yield

$$\int (\mathbf{E}_n \nabla_\perp \mathbf{E}_{n_0} - \mathbf{E}_{n_0} \nabla_\perp \mathbf{E}_n)\, dS = (k_n^2 - k_{n_0}^2) \int \mathbf{E}_n \mathbf{E}_{n_0}\, dV, \tag{2}$$

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where the integral on the left hand side is over the surface of the bead, and $\nabla_\perp$ denotes the normal derivative directed into the bead. The integral on the right hand side is over the volume of the cavity excluding the bead. In the next step we write $\mathbf{E}_n = \mathbf{E}_{n_0} + \delta_n$ and determine δ_n such that all electromagnetic boundary conditions are obeyed. The result depends on the shape of the bead. For circular beads, as applied here, all necessary formulas can be found, e.g., in Chap. 16 of Ref. [16]. The result is

$$\frac{k_n^2 - k_{n_0}^2}{k_n^2} = 2\,\frac{\Delta\nu_n}{\nu_n} = \frac{3}{2}\,\frac{V_R}{\int |\mathbf{E}_{n_0}|^2\,dV}\,(-2|\mathbf{E}_{n_0}|^2 + |\mathbf{B}_{n_0}|^2) \tag{3}$$

in accordance with Eq. (14) in Ref. [15]. Here $V_R = \frac{4\pi}{3}R^3$ is the volume of the bead and $\int(\mathbf{E}_{n_0})^2\,dV$ is the total electromagnetic energy in the resonator. For spherical beads thus only the field combination $-2|\mathbf{E}|^2 + |\mathbf{B}|^2$ can be obtained from the frequency shift $\Delta\nu$. Other combinations are in principle available by using different bead shapes.

In this way resonance depths a_n and frequency shifts $\Delta\nu_n$ were measured for about 200 eigenfrequencies in the range from 2 to 10 GHz. In the discussion of the resonance depth we can be short. The a_n are proportional to $E_{n\|}^2$, where $E_{n\|}$ is the component of the electric field parallel to the antenna at the antenna position [17]. Here we found a Porter-Thomas distribution for the resonance depths just as in two dimensions [18]; see Fig. 2. This shows that the $E_{n\|}$ are Gaussian distributed and uncorrelated for different eigenfrequencies. It should be noted that the situation is different if the distribution of $E_{n\|}^2$, or $|\Psi_n|^2$ in quantummechanical systems, is studied as a function of position for one single resonance. For this case significant deviations from Porter-Thomas distributions have been found for scarred and localized field distributions [19,20].

The main concern of this paper is the frequency shift $\Delta\nu$ as a function of position for the individual eigenfrequencies. Figure 1 shows three typical examples. The shaded surfaces correspond to surfaces of constant frequency shifts, i.e., constant $-2|\mathbf{E}|^2 + |\mathbf{B}|^2$. The $\Delta\nu$ values associated with the shaded surfaces are marked in Fig. 3 where the distribution function $P(\Delta\nu)$ of frequency shifts for the three eigenfrequencies are plotted. Though the measurement does not yield the field components separately but only a combination, nevertheless Fig. 1 gives a suggestive impression of the field distributions. Obviously the eigenfrequency shown in Fig. 1(a) corresponds to a standing wave between two parallel faces of the billiard. It can thus be considered as a three-dimensional analog of a bouncing-ball wave function. The field distribution shown in Fig. 1(b), on the other hand, has all features of a scar associated with a periodic orbit of the shape of a diamond. There are large frequency shifts along the orbit, and small or no shifts elsewhere. To corroborate the scar hypothesis a quantitative criterion, as it was developed by Agam and Fishman [21] for the quantum mechanical case, would be desirable. Unfortunately, however, such a criterion does not yet exist for three-dimensional electromagnetic cavities. Figure 1(c) finally shows a chaotically looking field distribution with no obvious association with a periodic orbit.

To get a quantitative tool to discriminate between chaotic and regular field distributions, we take as a hypothesis that for chaotic distributions all six field components are uncorrelated and Gaussian distributed.

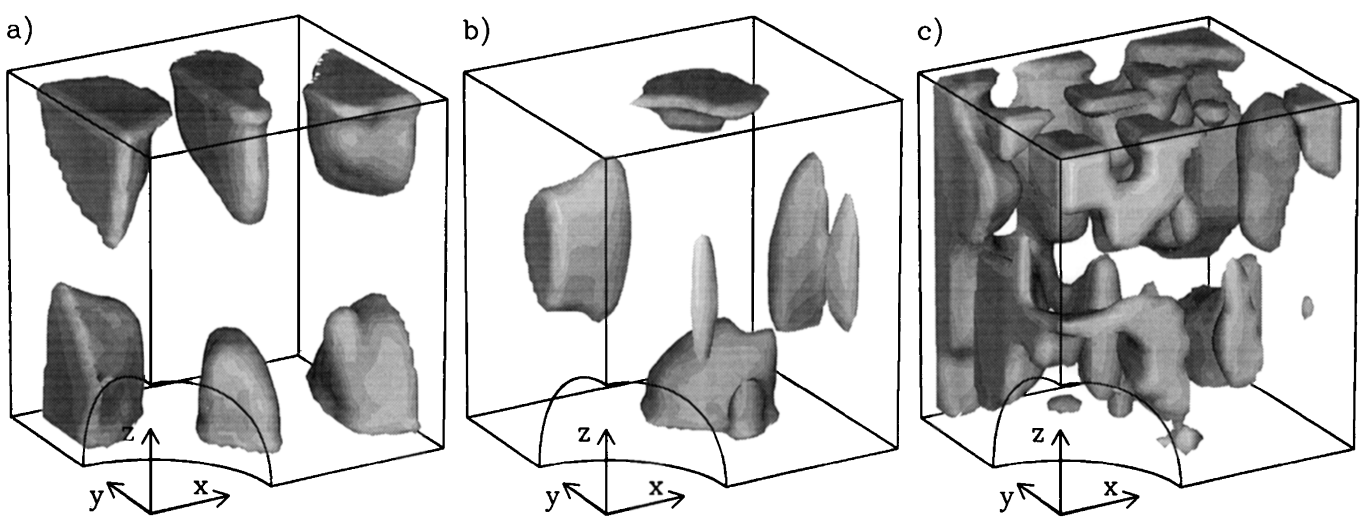

FIG. 1. Electromagnetic field distributions in a three-dimensional Sinai microwave billiard for three eigenfrequencies showing bouncing-ball (a), scarred (b), and chaotic (c) field distributions. The shaded surfaces correspond to points of constant frequency shift $\Delta\nu \sim -2\mathbf{E}^2 + \mathbf{B}^2$ (see text). The eigenfrequencies of the unshifted modes are 5.208, 2.897, and 8.293 GHz.

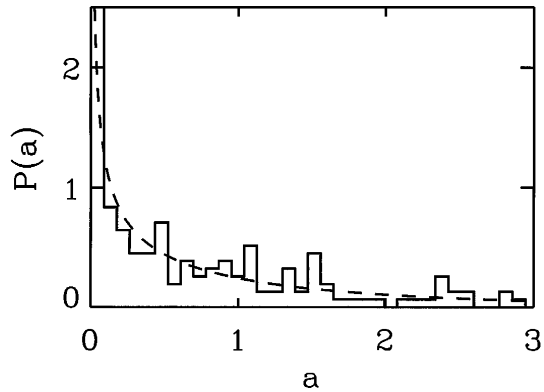

FIG. 2. Distribution of resonance depths. The dashed line corresponds to a Porter-Thomas distribution.

Then the distribution function $P(\Delta\nu)$ should be given by

$$P(\Delta\nu) = \int \delta(\Delta\nu + 2|\mathbf{E}|^2 - |\mathbf{B}|^2) \times \prod_{i=0}^{3} p(E_i)\, dE_i\, p(B_i)\, dB_i \qquad (4)$$

where

$$p(x) = \sqrt{a/\pi}\, e^{-\alpha x^2}. \qquad (5)$$

The integrations are easily carried out with the result

$$P(\Delta\nu) = \frac{\sqrt{2}\,\alpha^2}{3\pi}\, |\Delta\nu| \exp\left(-\frac{\alpha}{4}\,\Delta\nu\right) K_1\left(\frac{3}{4}\,\alpha|\Delta\nu|\right), \qquad (6)$$

where $K_1(x)$ is a modified Bessel function. Equation (6) may be considered a generalized Porter-Thomas distribution for the case of electromagnetic eigenfrequency field distributions. Figure 3 shows the experimental frequency-shift distributions, where the solid line corresponds to the expected behavior (6) for chaotic field distributions. For the comparison the first three moments of experimental and theoretical curves were adjusted to each other.

An inspection of the figure shows that for the chaotically looking field distribution in Fig. 1(c) the obtained $P(\Delta\nu)$ is described perfectly well by Eq. (6). This shows that the hypothesis of uncorrelated Gaussian distributed field components is correct, irrespective of their strong mutual coupling via the Maxwell equations. For the definitely nonchaotic field patterns depicted in Figs. 1(a) and 1(b), on the other hand, the found $\Delta\nu$ distribution deviates, not surprisingly, significantly from the behavior described by Eq. (6). A similar situation is found for eigenfunctions of the stadium billiard, where bouncing balls and scars lead to deviations from the otherwise observed Gaussian distributions of the wave function amplitudes [3]. Equation (6) is therefore indeed suited to check whether a field distribution is chaotic or not. By this we found that about 45% of the eigenfrequency distributions are chaotic with a significance of 68% with a tendency for this fraction to increase with increasing frequency.

The range of validity of the distribution function (6) must end, even for chaotic distributions, close to the surfaces of the resonator. Here the boundary conditions demand that the electric field vector is normal, and the magnetic field vector is parallel to the surfaces, i.e., there are now only three independent field components. Performing again the integrations in Eq. (4), but now only over one E and two B components, one obtains the frequency shift distribution

$$P(\Delta\nu) = \frac{\alpha}{2\sqrt{3}}\, e^{-\alpha\Delta\nu}\left[1 - \Theta(-\Delta\nu)\,\mathrm{erf}\left(\sqrt{\frac{3\alpha|\Delta\nu|}{2}}\right)\right], \qquad (7)$$

which should hold for bead positions close to the surface. Here $\Theta(x)$ is the Heaviside step function.

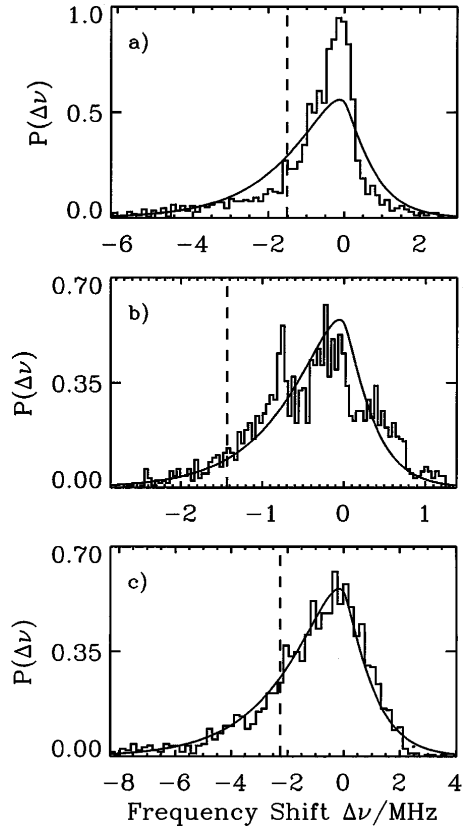

FIG. 3. Frequency shift distributions for the three eigenmodes shown in Fig. 1. The solid line corresponds to the theoretical expectation for chaotic field distributions. The frequency shifts corresponding to the shaded surfaces in Fig. 1 are marked by vertical dashed lines.

To check this prediction we took all field distributions found to be chaotic with the help of the above criterion and calculated the frequency shift distributions of the surface elements alone. Figure 4 shows the obtained histograms for three frequency regions, where the thickness of the surface was fixed to $d = 10$ mm. As the surface thickness should be properly expressed in units of the wavelength λ, the effective thickness increases with frequency, and one expects thus with increasing frequency a gradual change from the surface to the volume distribution. Exactly this behavior is found. In Fig. 4(a)

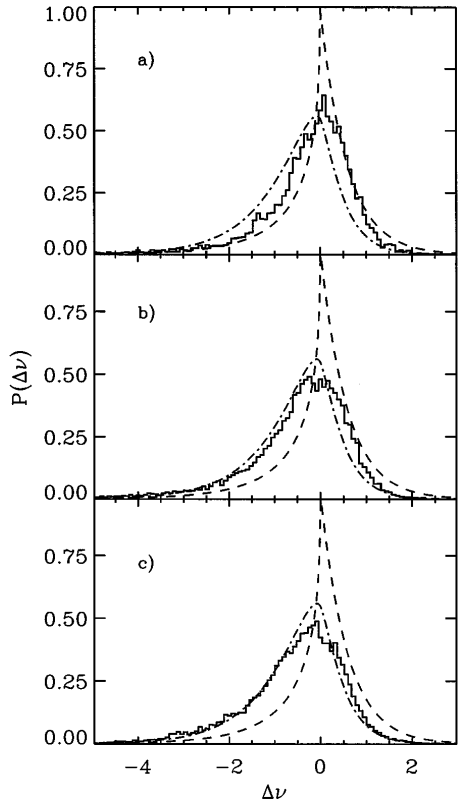

FIG. 4. Frequency shift distribution function $P(\Delta\nu)$ for bead positions closer than 10 mm to the resonator walls. The histograms were obtained by superimposing the results from all eigenfrequencies with chaotic field distribution in the range (a) 3–5.5 GHz, (b) 5.5–7 GHz, and (c) 7–10 GHz. The dashed and the dash-dotted lines correspond to the theoretical expectations for the surface and the volume, respectively.

corresponding to d/λ in the range 0.1 to 0.18 the experimental distribution is rather close to the surface distribution (7), though the sharp cusp predicted by theory is not found in the experiment. In Fig. 4(c) with d/λ in the range 0.23 to 0.33 the found distribution approaches already the volume distribution (6). An analogous behavior is found upon variation of d (not shown).

We have shown that the perturbing bead method is well suited to obtain information on electromagnetic field distributions in three-dimensional microwave cavities. For the chaotic field distributions all field components were found to be uncorrelated Gaussian distributed in spite of their strong coupling via the Maxwell equations. This has been demonstrated on three levels of increasing complexity. The distribution of resonance depths shows that the **E** component parallel to the antenna wire is Gaussian distributed. From the $\Delta\nu$ distribution in the surface one concludes that the **E** component perpendicular to the surface and the two **B** components in the surface are uncorrelated Gaussian distributed, whereas from the $\Delta\nu$ distribution in the volume the same is found for altogether six **E** and **B** components. One may wonder whether these findings can be understood as for scalar fields [22] by assuming that the field components at one point can be described by a random superposition of plane waves. This is indeed the case. It is rather simple to show with the help of the central limit theorem that a superposition of plane electromagnetic waves entering from random directions with randomly distributed amplitudes yields exactly the experimentally found behavior.

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