We also wish to thank F. O. Miesterfeld for assisting in preparation of the sample.

[1]C. Kittel, Phys. Rev. 110, 836 (1958).
[2]K. B. Vlasov and B. Kh. Ishmukhametov, J. Exptl. Theoret. Phys. (U.S.S.R.) 37, 725 (1959) [translation: Soviet Phys.-JETP 37, 531 (1960)].
[3]E. G. Spencer, R. T. Denton, and R. P. Chambers, Phys. Rev. 125, 1950 (1962).
[4]Calculated from values of the saturation magnetostriction determined by R. Hall and P. Bailey. We are indebted to J. A. Osborn of Westinghouse Research Laboratories for communicating their results.
[5]E. H. Turner, in Advances in Quantum Electronics 1961 (Columbia University Press, New York, 1961), p. 427.

# SOLUTION OF THE GAP EQUATION FOR A SUPERCONDUCTOR

G. J. Culler, B. D. Fried, R. W. Huff,* and J. R. Schrieffer†
RW Division, Thompson Ramo Wooldridge, Inc., Canoga Park, California
(Received April 13, 1962)

A central problem in comparing the Bardeen-Cooper-Schrieffer theory of superconductivity with experiment is the construction and solution of an integral equation giving the energy gap parameter $\Delta(E)$ as a function of the excitation energy $E$. The properties of superconductors, which are insensitive to the energy dependence of $\Delta(E)$, are well accounted for by approximating the kernel of the equation by a product (or sum of several products) of step functions in momentum space.[1] More generally, the kernel must explicitly include the dynamic interaction between electrons due to exchange of phonons and the (screened) Coulomb interaction. Using a finite temperature generalization of the Green's function formalism of Nambu,[2] Schrieffer and Wilkins[3] have derived an energy gap equation including phonon and Coulomb interactions. We present here the results of an exact numerical solution of this energy gap equation achieved with the use of a novel "on-line" computational scheme.[4]

If we use a spherical band model for electrons which are coupled by deformation potential matrix elements to phonons with a Debye spectrum, the $s$-wave part of the energy gap equation can be reduced to:

$$\Delta(E)=k\int_0^{\omega_1}d\epsilon'\frac{\Delta(E')}{E'}$$

$$\times\{F(E'+E)+F(E'-E)-C\}\tanh\left(\frac{E'}{2k_BT}\right), \qquad (1)$$

where

$$E'=[\epsilon'^2+\Delta^2(E')]^{1/2}, \qquad (2)$$

and

$$F(x)=\tfrac{1}{2}-x+x^2\log\left|\frac{x+1}{x}\right|, \qquad (3)$$

all energies being measured in units of the Debye energy, $k_B\theta_D$. Here $k_B$ is Boltzmann's constant and $T$ is the absolute temperature. The "coupling constant" $k$ is proportional to the square of the deformation potential constant and gives the strength of the phonon part of the interaction. The parameter $kC$ is a pseudopotential for the screened Coulomb interaction defined to include interactions involving states of energy greater than $\omega_1$[5,6]; the results are independent of $\omega_1$ if $\omega_1\gtrsim 10$. While, in principle, $kC$ can be estimated from first principles, we choose it to be an adjustable parameter. The phonon part of the kernel, aside from the tanh factor, is of the form derived by Eliashberg[7] at zero temperature. We have neglected damping effects as well as corrections to the phonon interaction which are formally of order $k$ but which are negligible for weak coupling superconductors and of importance only near the transition temperature for strong coupling cases.[3]

An energy gap equation employing a nonseparable square well approximation to the Bardeen-Pines phonon interaction has been solved numerically by Swihart[8] through a quasi-linearization of the integral equation. While our results are in many respects similar to those of Swihart, there are important differences.

The solution of Eq. (1) was carried out using an experimental console which allows direct control of the computer and also provides immediate graphical representations of the results of each significant computational step.[4] The principal results of our calculations are summarized in Figs. 1 through 4. In Fig. 1, the

relation between the coupling constant $k$ and $\Delta_0 \equiv \Delta(E)|_{E=0}$ is shown for $T=0$ and several values of the Coulomb pseudopotential strength $C$. The true energy gap, $2\Delta(\Delta)$, is approximately $2\Delta_0$, since $\Delta$ varies slowly with $E$ for $0<E<\Delta$ in all cases of interest (see Figs. 2 and 3). Also shown in Fig. 1 are the results corresponding to $C=0$ for a separable potential of strength $k$ for $0<\epsilon<\omega=0.436$ and zero otherwise, giving

$$\Delta_0 = \omega/\sinh(1/k). \tag{4}$$

For the $C\neq 0$ case, $k$ is replaced by an effective coupling constant $\hat{k}(k,C)$.

The critical temperature $T_c$ is the maximum value of $T$ for which Eq. (1) has a nontrivial solution with given $k$ and $C$. The ratio, $2\Delta_0(0)/k_B T_c$, is essentially 3.50 for $k \ll 1$, and decreases slightly for $k \geq 0.5$. Damping effects are expected to increase this ratio for strong coupling, since the temperature-dependent part of the damping will lower $T_c$ but not affect $\Delta_0(0)$. For $0<T<T_c$ the variation of $\Delta_0(T)$ with $T$ is so accurately de-

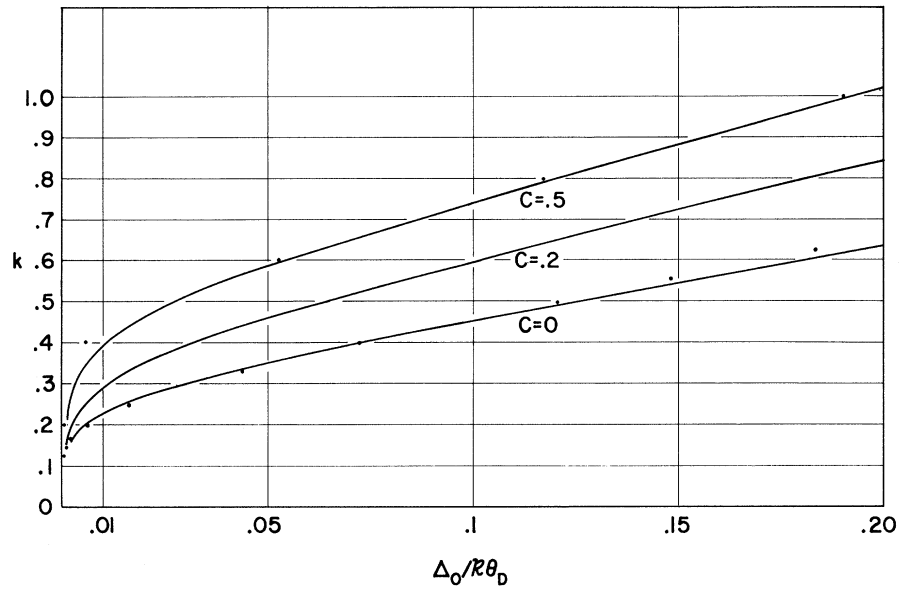


FIG. 1. Coupling constant, $k$, vs energy gap, $\Delta_0$, for $T=0$ and $C=0$, 0.2, and 0.5. Results obtained by solving Eq. (2) are shown as solid curves, while the heavy points were computed using the separable kernel approximation described in the text for $C=0$ and $C=0.5$.

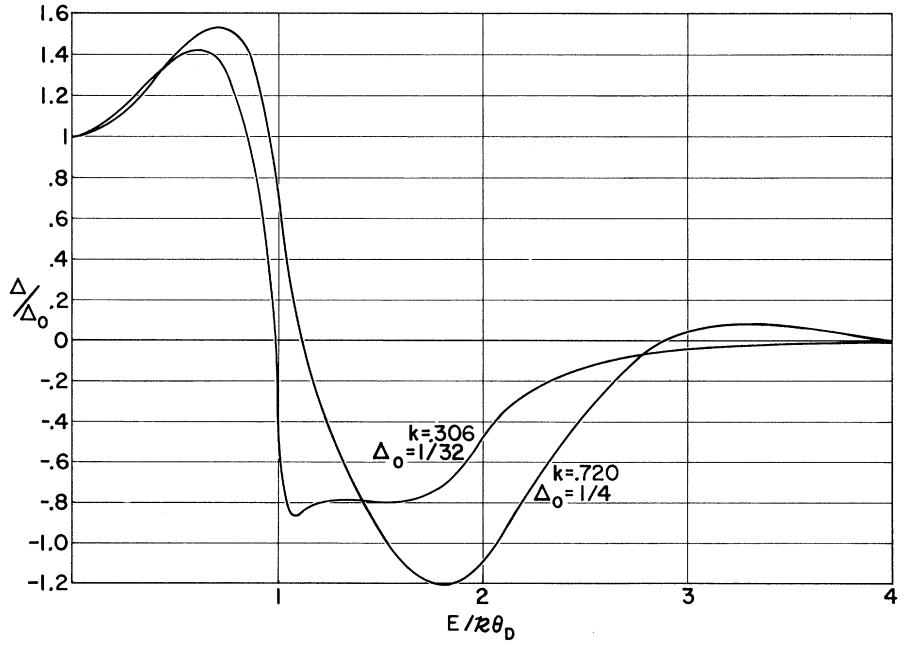


FIG. 2. Energy gap, $\Delta$, vs energy for $T=0$, $C=0$, in the weak-coupling ($\Delta_0=\frac{1}{32}$) and strong-coupling ($\Delta_0=\frac{1}{4}$) regions.

scribed by

$$\Delta_0(T)/\Delta_0(0) = \tanh[T_c\Delta_0(T)/T\Delta_0(0)] \qquad (5)$$

as to be not worth plotting, a result noted by Swihart[8] and in agreement with the strong coupling limit of the separable potential approximation.

In Fig. 2, we show the actual shape of the solutions $\Delta(E)$ of Eq. (1) for $T=0$ and $C=0$ in the weak and strong coupling limits. While the curves are similar to Swihart's in having a negative minimum followed in the strong coupling case by a small positive maximum, the low-energy behavior is distinctly different. In all cases we find that $\Delta(E)$ increases for small $E$, passing through a maximum before changing sign at $E \sim 1$. Swihart, on the other hand, found in all cases that $\Delta(E)$ decreased monotonically toward the negative minimum and conjectured that this behavior would obtain for all physically rea-

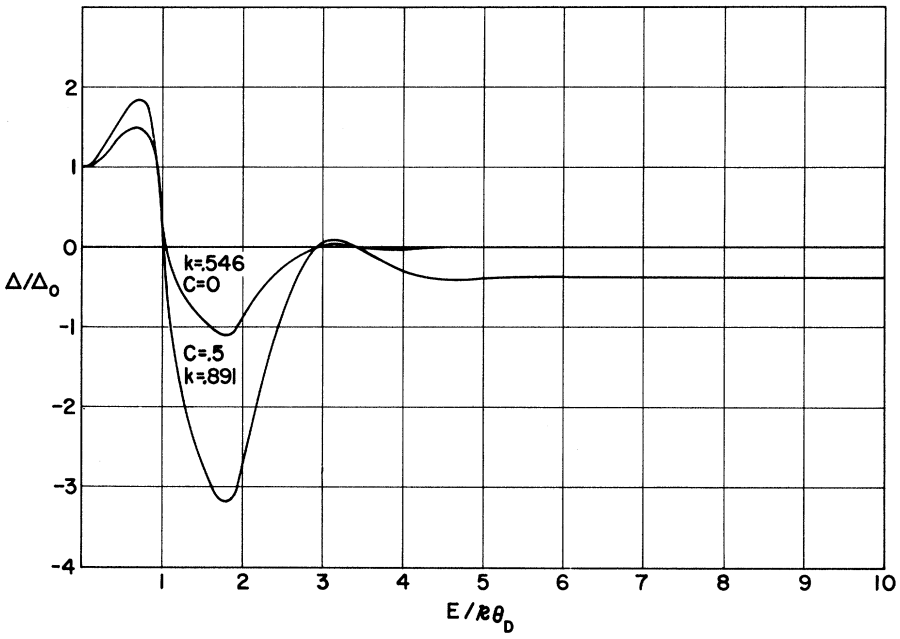


FIG. 3. Energy gap, $\Delta$, vs energy for $T=0$ in the strong-coupling region ($\Delta_0=0.15$) with $C=0$ and $C=0.5$.

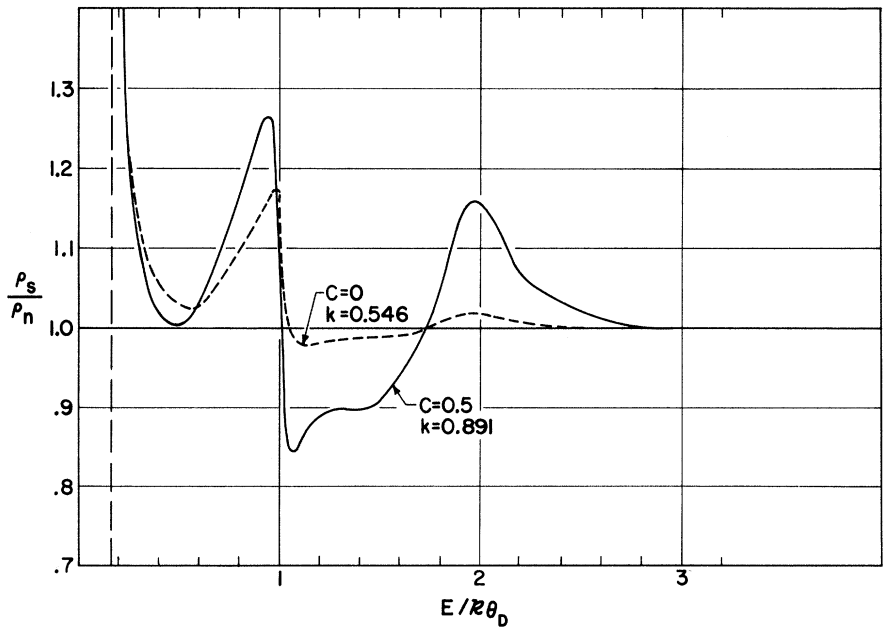


FIG. 4. Ratio of superconducting and normal density-of-state factors for $T=0$, $\Delta_0=0.15$ with $C=0$ and $C=0.5$.

sonable kernels. On physical grounds we expect the kernel to exhibit a damped resonance near the Debye energy. Thus we feel the initial rise and maximum in $\Delta(E)$ is a true physical effect.[9]

For $T \neq 0$ we find $\Delta(E, T)/\Delta_0(T)$ is almost independent of $T$, provided $k$ and $C$ are held fixed. The effects of $C \neq 0$ are shown in Fig. 3 for $T = 0$ and $\Delta_0 = 0.15$, a value roughly appropriate to lead.

The ratio $\rho_s/\rho_n$ of the density of states for the superconducting and normal states follows directly from $\Delta(E)$ if we assume that $\rho_n$ is a constant:

$$\rho_s/\rho_n = \frac{d\epsilon}{dE} = [1 - (\Delta(E)/E)^2]^{-1/2}\left[1 - \frac{\Delta(E)}{E}\frac{d\Delta(E)}{dE}\right]. \quad (6)$$

The ratio is of particular interest since it can be measured directly from electron tunneling experiments.[10] The functions $\rho_s(E)/\rho_n(E)$ corresponding to the curves of Fig. 3 are shown in Fig. 4, the solid curve corresponding roughly to the case of lead at low reduced temperature. Recent tunneling experiments by Giaever, Hart, and Megerle[11] indicate that $\rho_s/\rho_n - 1$ has a more complicated form than the positive quantity $E/\epsilon - 1$ predicted by the separable-potential approximation. They find at least one zero crossing of the function and most likely peaks occur in $\rho_s$ near the Debye energy.[12] Detailed comparison of our results with experiment has not yet been attempted.

The inclusion of finite lifetime effects is expected to smooth out the energy dependence of $\Delta$, particularly near the Debye energy. This will tend to decrease the predicted peaks in the density-of-states curves. The predicted maximum in $\Delta(E)$ is expected to affect thermodynamic as well as transport properties of superconductors.

We wish to thank Dr. I. Giaever, Dr. H. R. Hart, and Dr. K. Megerle for communicating their results to us prior to publication.

---

*Permanent address: Lawrence Radiation Laboratory, University of California, Berkeley, California.

†Permanent address: University of Illinois, Urbana, Illinois.

[1]J. Bardeen and J. R. Schrieffer, Progress in Low-Temperature Physics, edited by J. C. Gorter (North-Holland Publishing Company, Amsterdam, 1961), Vol. III, p. 170.

[2]Y. Nambu, Phys. Rev. 117, 648 (1960).

[3]J. R. Schrieffer and J. W. Wilkins (to be published).

[4]G. J. Culler and R. W. Huff, Proceedings of the Western Joint Computer Conference, May, 1962 (unpublished).

[5]N. N. Bogoliubov and V. V. Tolmachev, and D. V. Shirkov, A New Method in the Theory of Superconductivity (Consultants Bureau, Inc., New York, 1959), pp. 80-84.

[6]P. W. Anderson and P. Morel, Phys. Rev. Letters 5, 136 (1960).

[7]G. M. Eliashberg, J. Exptl. Theoret. Phys. (U.S.S.R.) 38, 966 (1960) [translation: Soviet Phys.-JETP 11, 696 (1960)].

[8]J. C. Swihart, IBM J. Research Develop. 6, 14 (1962).

[9]This is in agreement with the results of P. Morel and P. W. Anderson, Phys. Rev. 125, 1263 (1962).

[10]I. Giaever, Phys. Rev. Letters 5, 147 (1960).

[11]I. Giaever, H. R. Hart, Jr., and K. Megerle, Phys. Rev. 126, 941 (1962).

[12]I. Giaever (private communication).

---

## PHOTOEMISSION OF HOLES FROM TIN INTO GALLIUM ARSENIDE

Richard Williams
RCA Laboratories, Princeton, New Jersey
(Received April 16, 1962)

Several authors have reported experiments in which they observed photoemission of electrons from metals into semiconductors.[1-4] Photoexcited electrons may be collected at a blocking contact between the metal and an $n$-type semiconductor. Such a contact enables an experimental separation to be made between photoexcited electrons in the metal and those at the Fermi level. By using a blocking contact of a metal to a $p$-type semiconductor it should be possible, using the same principle, to observe photoemission of holes from the metal into the semiconductor. The theoretical treatment of Quinn[5] indicates that for small excitation energies around 1 eV the mean free path for energy loss in a metal may be the same order of magnitude for both holes and electrons. In the present note, experimental evidence is presented which is believed to demonstrate the photoemission of holes from tin into $p$-type gallium arsenide.

Measurements were made of the spectral de-