HIGH-ENERGY NUCLEAR SCATTERING AND REGGE POLES*

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It has been suggested[1-3] that Regge pole terms dominate the high-energy behavior of two-body scattering amplitudes involving pions, nucleons, etc. It is possible[4,5] to interpret the available data on π-N, N-N, $\bar{N}$-N, and K-N scattering in terms of the Regge pole hypothesis, with the assumption that all processes in which the quantum numbers of the vacuum can be exchanged are dominated at high energies by a particular term. This "Pomeranchuk-Regge term" is supposed to have $\alpha_P(0)=1$, where $\alpha_P(t)$ is the position of the Regge pole in the angular momentum plane as a function of t, the exchanged mass squared. Any elastic scattering amplitude,[6] say for particles X and Y, has the form

$$T_{XY}(s,t)=2s_0\left(\frac{s}{s_0}\right)^{\alpha_P(t)} b_{XPY}(t)\,\frac{1+e^{-i\pi\alpha_P(t)}}{2\sin\pi\alpha_P(t)} \tag{1}$$

at high energies near the forward direction. Here s is the total mass squared in the reaction $X+Y \to X+Y$ and s_0 is an arbitrary parameter with the same dimensions. The total cross section at high energies is

$$\sigma_{XY}=-\mathrm{Im}T_{XY}(s,0)/s=b_{XPY}(0),$$

a constant, because of the assumption that $\alpha_P(0)=1$. In fact σ_{XY} must be the product[7] of a factor depending on X times a factor depending on Y:

$$\sigma_{XY}=\sigma_{XX}^{1/2}\sigma_{YY}^{1/2}. \tag{2}$$

When the energy is not quite so high, we must include in the scattering amplitude terms corresponding to other Regge poles, associated with ρ, ω, etc. In this way, one can understand and correlate[5] the observed deviations from constant total cross sections in the region of 5-20 Bev. For simplicity, we shall concentrate on the dominant Pomeranchuk term. We now drop the indices P, X, and Y.

The angular distribution in the diffraction scattering peak can be read off from Eq. (1). Knowing that α increases with t, which is negative for physical momentum transfers, we see that the diffraction peak becomes narrower with increasing energy. Putting

$$B(t)\equiv b(t)i\{1+\exp[-i\pi\alpha(t)]\}[b(0)]^{-1}[\sin\pi\alpha(t)]^{-1},$$

we have in general,

$$T(s,t)=-i\sigma sB(t)(s/s_0)^{\alpha(t)-1}. \tag{3}$$

A rough description can be obtained if, for a suitably chosen s_0, we suppose that $B(t)$ varies slowly compared with $(s/s_0)^{\alpha(t)}$. Then for small t, setting $\epsilon\equiv\alpha'(0)$, we have $\alpha-1\approx\epsilon t$ and $B(t)\approx 1$. We thus obtain the "exponential approximation,"

$$T(s,t)\approx -i\sigma s\exp[\epsilon t\ln(s/s_0)], \tag{4}$$

that clearly exhibits the shrinking of the diffraction peak. The CERN data on p-p scattering seem consistent[4] with Eq. (4).

The shrinking corresponds to an increase of radius and of transparency such that the total cross section remains constant. Mathematically, we may put

$$T=-8\pi i\sum_l(2l+1)P_l(1+t/2s)(1-e^{2i\delta_l}) \tag{5}$$

at high energies. We turn the sum over l into an integral, introduce the impact parameter $D=2ls^{-1/2}$, and use $P_l(1+t/2s)\approx J_0(D\sqrt{-t})$. We then calculate $F(s,t)$, an amplitude such that $d\sigma/dt=|F(s,t)|^2$. For N-N scattering at high energies we have $F=(-4s\sqrt{\pi})^{-1}T$ and so

$$F=\frac{i}{2\sqrt{\pi}}\int_0^\infty 2\pi D\,dD\,[1-S_D(s)]J_0(D\sqrt{-t})$$

$$=\frac{i}{2\sqrt{\pi}}\int d^2D\,[1-S_D(s)]\exp(i\vec{D}\cdot\vec{q}), \tag{6}$$

where $S_D(s)=e^{2i\delta_l}$ and $q^2=-t$. Now, using the inverse Fourier transform, we may employ either the exact asymptotic expression (3) or the approximate expression (4) to obtain the absorption coefficient $1 - S_D(s)$ for impact parameter D:

$$1 - S_D(s) = \frac{\sigma}{8\pi^2}\int d^2q\,(s/s_0)^{\alpha(-q^2)-1}B(-q^2)\exp(i\vec{D}\cdot\vec{q}), \tag{7}$$

$$1 - S_D(s) \approx \frac{\sigma}{8\pi^2}\int d^2q\,\exp[-\epsilon q^2\ln(s/s_0)]\exp(i\vec{D}\cdot\vec{q})$$
$$= \frac{\sigma}{8\pi}[\epsilon\ln(s/s_0)]^{-1}\exp\{-D^2[4\epsilon\ln(s/s_0)]^{-1}\}. \tag{8}$$

The logarithmic increase of radius squared and of transparency is obvious. The elastic scattering cross section, using the approximation of Eq. (8), is just

$$\sigma_{\mathrm{el}} = \int d^2D\,|1 - S_D(s)|^2 \approx \frac{\sigma^2}{32\pi\epsilon\ln(s/s_0)}, \tag{9}$$

which tends to zero as $s\to\infty$.

So far, we have discussed only the scattering of particles which have no "anomalous thresholds." Now we turn to systems with anomalous thresholds, like nuclei. Such a system, with mass M, can dissociate virtually into two parts with masses M_1 and M_2, where $(M_1+M_2)^2 > M^2 > M_1^2 + M_2^2$. More concretely, we may say that the system with anomalous thresholds has a wave function which extends out in space a distance L, where L^2 is greater than the sum of the squares of the Compton wavelengths of the parts. A system with very prominent anomalous thresholds, like a nucleus, can be treated approximately as a composite system described by a wave function referring to the coordinates of the component parts.

Using such a description for a nucleus and assuming the Regge pole hypothesis for N-N scattering, let us compute the properties of high-energy N-nucleus scattering. (The same considerations will of course apply to nuclear scattering of π and K mesons.)

We employ the semiclassical ray method, which should be valid in our energy range, $\gg 1$ Bev. We ignore the velocities of the particles in the nucleus, so that s is always computed for the incoming nucleon and any one target nucleon considered as being at rest in the nucleus. The probability distribution $|\psi|^2$ of the nucleon positions is integrated over the z coordinates so as to give a probability distribution $P(\vec{\rho}_1, \vec{\rho}_2, \cdots)$ of two-dimensional vectors $\vec{\rho}_1$, $i = 1, \cdots A$. Then the transmission coefficient $S_D^A(s)$ for the nucleus is just the averaged product of the transmission coefficients for the individual nucleons:

$$S_D^A(s) = \int\cdots\int d^2\rho_1\cdots d^2\rho_A\,P(\vec{\rho}_1,\vec{\rho}_2,\cdots)\times\prod_{i=1}^{A} S_{|\vec{D}-\vec{\rho}_i|}(s). \tag{10}$$

This expression can then be substituted into a formula analogous to Eq. (6) to give the total cross section and diffraction peak in N-nucleus collisions.

As a simple example, let us take the deuteron. We put the center of mass at the origin so that $\vec{\rho}_1 = \vec{\rho}/2$ and $\vec{\rho}_2 = -\vec{\rho}/2$, where $\vec{\rho}$ is the two-dimensional relative coordinate. Let the wave function (ignoring spin) be $\psi(\vec{\rho}, z)$ and put

$$G(p^2) = \int_{-\infty}^{\infty}dz\int d^2\rho\,|\psi|^2\exp(i\vec{p}\cdot\vec{\rho}). \tag{11}$$

Then the scattering amplitude and total cross section, with the use of Eqs. (6), (7), and (10), come out:

$$F^A = \frac{i}{4\sqrt{\pi}}\Big\{2\sigma G(-t/4)B(t)(s/s_0)^{\alpha(t)-1} - \frac{\sigma^2}{8\pi^2}\int d^2p\,G(p^2)B(-(\vec{q}/2-\vec{p})^2)$$
$$\times B(-(\vec{q}/2+\vec{p})^2)(s/s_0)^{\alpha(-(\vec{q}/2-\vec{p})^2)+\alpha(-(\vec{q}/2+\vec{p})^2)-2}\Big\}, \tag{12}$$

$$\sigma^A = 2\sigma - \frac{\sigma^2}{8\pi^2}\mathrm{Re}\int d^2p\,G(p^2)[B(-p^2)]^2(s/s_0)^{2\alpha(-p^2)-2}. \tag{13}$$

We note that besides the Pomeranchuk-Regge pole term (with coefficient, in the forward direction, twice as large as in N-N scattering), there is an eclipse term in the form of a continuous line of Regge poles. These form a cut along the real axis of the angular momentum plane extending up to a point that approaches unity from below, like α itself, when t approaches zero from negative values. At very high energies, the eclipse term, at $t=0$, vanishes like $[\ln(s/s_0)]^{-1}$ and we are left with a total cross section equal to 2σ. That is clearly correct, since each nucleon has become highly transparent. At moderate energies of a few Bev, the eclipse term is just the one studied by Glauber.[8]

Now, let us treat a heavy nucleus, assuming that the nucleons are independent particles bound to a center at the origin. From Eq. (10) we obtain

$$S_D^A(s) = \prod_{i=1}^{A} \int d^2\vec{\rho}_i P_i(\vec{\rho}_i) S_{|\vec{D}-\vec{\rho}_i|}(s),$$

and, using Eq. (7),

$$1 - S_D^A(s) = 1 - \prod_{i=1}^{A}\left(1 - \int d^2\rho_i P_i(\vec{\rho}_i)\frac{\sigma}{8\pi^2}\int d^2q\{\exp[i\vec{q}\cdot(\vec{D}-\vec{\rho}_i)]\}B(-q^2)(s/s_0)^{\alpha(-q^2)-1}\right). \quad (14)$$

With $\sigma \approx 40$ mb experimentally, we note that for a nucleus of any size the individual absorption coefficients τ_i are less than $\frac{1}{2}\sigma P_i(0) \ll 1$, so that we may replace $1-\tau_i$ by $e^{-\tau_i}$, obtaining

$$1 - S_D^A(s) = 1 - \exp[-\tau_D^A(s)], \quad (15)$$

$$\tau_D^A(s) = \int d^2\rho\left[\sum_i P_i(\rho)\right]\frac{\sigma}{8\pi^2}\int d^2q\{\exp[i\vec{q}\cdot(\vec{D}\cdot\vec{\rho})]\} \times B(-q^2)(s/s_0)^{\alpha(-q^2)-1}. \quad (16)$$

The total cross section is just

$$\sigma^A(s) = 2\int d^2D\left\{1 - \mathrm{Re}\exp\left[-\tau_D^A(s)\right]\right\}, \quad (17)$$

while the absorption cross section is easily seen to be given by the formula

$$\sigma_{\mathrm{abs}}^A(s) = \int d^2D\left\{1 - \left|\exp\left[-2\tau_D^A(s)\right]\right|\right\}. \quad (18)$$

For actual evaluation, let us make the exponential approximation (4) for the N-N scattering amplitude, putting $B=1$ and $\alpha = 1+\epsilon t$. For the nucleon density, we use a uniform model with a sharp cutoff at radius R_A, which gives

$$\sum_i P_i(\rho) = A(\tfrac{4}{3}\pi R_A^3)^{-1} 2(R_A^2-\rho^2)^{1/2}\eta(R_A^2-\rho^2). \quad (19)$$

Putting $\bar{D} = D/R_A$ and $\beta^2 = [4\pi\epsilon\ln(s/s_0)](\pi R_A^2)^{-1}$, we obtain

$$\sigma^A(s) = 2\pi R_A^2\int_0^\infty 2\bar{D}d\bar{D}\left\{1 - \exp\left[-\sigma A\left(2\pi R_A^2\right)^{-1}\phi(\bar{D}^2,\beta^2)\right]\right\}, \quad (20)$$

$$\sigma_{\mathrm{abs}}^A(s) = \pi R_A^2\int_0^\infty 2\bar{D}d\bar{D}\left\{1 - \exp\left[-\sigma A\left(\pi R_A^2\right)^{-1}\phi(\bar{D}^2,\beta^2)\right]\right\}, \quad (21)$$

with

$$\phi(\bar{D}^2,\beta^2) = [8\pi\epsilon R_A\ln(s/s_0)]^{-1}\int d^2\rho\, 3(R_A^2-\rho^2)^{1/2}\eta(R_A^2-\rho^2)\exp\{-(D-\rho)^2/[4\epsilon\ln(s/s_0)]\}$$

$$= 3\beta^{-2}[\exp(-\bar{D}^2\beta^{-2})]\int_0^1 \bar{\rho}d\bar{\rho}\,(1-\bar{\rho}^2)^{1/2}[\exp(-\bar{\rho}^2\beta^{-2})]I_0(2\bar{D}\bar{\rho}\beta^{-2}). \quad (22)$$

At very high energies or large β^2, ϕ becomes very small everywhere $[\phi \approx \beta^{-2}\exp(-\bar{D}^2\beta^{-2})]$, so that the exponential may be expanded and the integral over $\bar{D}$ performed. Both cross sections then become just σA, as they must in the limit of infinite energy. At very low energy or small β^2, we have $\phi \approx \frac{3}{2}(1-\bar{D}^2)^{1/2}\eta(1-\bar{D}^2)$ and the cross sections become $2\pi R_A{}^2$ and $\pi R_A{}^2$, respectively, plus correction terms that vanish with $\pi R_A{}^2/\sigma A$. Again that is what we expect. The transition region may be investigated numerically.

For $\sigma A/\pi R_A{}^2 \approx (12)^{1/3}$ (nuclei around carbon), we have calculated $\sigma_{\text{abs}}{}^A$ at $\beta^2 = 0.05$, 0.15, 0.30, and 0.45. With the rough assignments $\epsilon \approx 1$ Bev^{-2} and $s_0 \approx 2$ Bev^2 of reference 4, these values of β^2 correspond to laboratory energies of 10, 10^3, 10^6, and 10^9 Bev, respectively. Of course the energy for a given β^2 is very sensitive to ϵ and s_0, which are badly known. The values of $\sigma_{\text{abs}}{}^A/\pi R^2$ come out to be 0.98, 1.10, 1.22, and 1.33, showing a very gradual approach to the asymptotic value of 2.29. Mathematically, the slowness of the approach to the high-energy constant cross section is associated with the appearance of a Regge cut instead of merely Regge poles.

For $\sigma A/\pi R_A{}^2 \approx (216)^{1/3}$ (nuclei around lead), we have calculated $\sigma_{\text{abs}}{}^A$ at $\beta^2 = 0.007$, 0.021, 0.042, and 0.063, corresponding roughly to the same energies given above for carbon. The values of $\sigma_{\text{abs}}{}^A/\pi R^2$ are 1.05, 1.14, 1.24, and 1.32, respectively. Here the asymptotic value of 6 is approached even more gradually.

In general, for a system of many nucleons, a sizable fractional increase of cross section requires the effective nucleon area to be comparable with the area of the system ($\beta^2 \approx 1$). Thus, even for a light nucleus, enormous energies are required for the increase to be easily observable. However, at the highest cosmic-ray energies, the effect may be of importance. At 10^6 Bev, treating air like carbon, we predict a 25% increase in the absorption cross section of air over the value at 10 Bev. It is interesting that Nikolskii et al.[9] have reported a change in the character of extensive air showers in the same energy region, indicating an increase in the absorption coefficient of air. The result has not been confirmed by other laboratories.

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