Belgrade, 1961 (unpublished), Paper NE109.

[2] Wells, Hooton, and Page, J. Brit. Inst. Radio Engrs. 20, 749 (1960).

[3] J. P. Elliott and B. H. Flowers, Proc. Roy. Soc. (London) A242, 57 (1957).

[4] G. E. Brown, L. Castillejo, and J. A. Evans, Nuclear Phys. 22, 1 (1961).

[5] C. Milone and A. Rubbino, Nuovo cimento 13, 1035 (1959); C. Milone, Phys. Rev. Letters 3, 43 (1959).

[6] Reported by L. Katz, *Proceedings of the International Conference on Nuclear Structure, Kingston, Canada* (University of Toronto Press, Toronto, 1960), p. 713.

[7] N. Tanner and G. Thomas, Proceedings of the Rutherford Conference, Manchester, England (to be published); S. G. Cohen, P. S. Fisher, and E. K. Warburton, Phys. Rev. 121, 858 (1961).

[8] A. S. Penfold and B. M. Spicer, Phys. Rev. 100, 1377 (1959).

[9] L. Katz, R. N. H. Haslam, R. J. Horsley, A. G. W. Cameron, and R. Montalbetti, Phys. Rev. 95, 464 (1954).

[10] K. H. Lokan (to be published).

BINDING ENERGY OF THE TRITON*

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The first purpose of this Letter is to correct an erroneous result published last year,[1] and the second is to give new results for potentials of considerable interest from the meson-theoretical point of view.

In recoding the calculation of the binding energy of the triton, and of He^3, for the IBM 7090, we discovered a coding mistake in the IBM-704 code, which had evaded our previous checks. After correcting this mistake, we have rechecked the whole code in a number of ways, with satisfactory results. Nevertheless, we would greatly welcome a completely independent triton calculation by another group. As far as we know, this is the only way to be really sure that a code of this magnitude is free of mistakes.

This correction alters the results given in reference 1 for the Gammel-Brueckner (GB) potential.[2] The corrected result is $E(H^3) \leq -5.7$ Mev, which is now higher than the known ground-state energy of the triton, $E(H^3) = -8.492$ Mev. Since a better trial function than ours would lower the theoretical energy, there is no longer an immediate contradiction. We do not know whether the theoretical energy can be pushed down to -8.492 Mev, but rather doubt it (see later).

With the faster speed of the IBM 7090 it proved possible to test a number of different potentials so as to get a clearer over-all picture. In addition to the GB potential, we have calculated with the following:

(1) The three potentials used by Feshbach and Pease[3] (FP), but not their two interpolated potentials. The FP potentials have Yukawa well shape and zero core radius. They fit all low-energy data, but do not fit modern high-energy data.

(2) The first of the three "Yukawa" potentials listed in Table II, p. 141, of Hu and Massey[4] (HM). This potential is similar to the FP potentials, but fails to fit the triplet effective range, which is about 50% too high. The other two "Yukawa" potentials of HM have even longer ranges. However, this potential has been used in a triton calculation by Hu and Hsu[5] and thus provides a useful comparison.

(3) The latest potential of Hamada and Johnston[6] (HJ).

(4) The preliminary potential of the Yale group[7] (Yale). These last two potentials are very similar to each other, and of considerable theoretical interest, since they have the one-pion exchange tail, are in qualitative agreement with meson-theoretical predictions in the intermediate region of distances, and give reasonable fits to all the available data on the nuclear two-body systems.

Our results for all these potentials are given in Table I, which we shall now discuss:

For the FP potentials, our energies are systematically about 1 Mev lower (better) than those of reference 3. This is not surprising, since we use a more elaborate wave function, and have done considerable minimizing to find best parameter values. Our wave function contained two exemplars of the principal ${}^2S_{1/2}$ state (one in reference 3), one function for state 3 of Derrick and Blatt[8] (omitted by FP), and the same number of D states as FP used. Furthermore, our functions[1] $u(r)$ and $v(r)$ were more flexible, both being of the form:

$$u(r) = \exp(-\gamma r)[1 + a_1 \exp(-\delta r) + a_2 \exp(-2\delta r) + a_3 \exp(-3\delta r)]. \tag{1}$$

The FP form is obtained by setting $a_1 = a_2 = a_3 = 0$. Thus, the fact that we could better the FP results by only about 1 Mev is a considerable success of the old calculation of reference 3.

The discrepancy between our result and that of Hu and Hsu[5] is larger (-8.1 compared to -6.4 Mev), and could be made larger still by additional parameter searching. It is possible that Hu and Hsu did not do as much minimizing on parameter values as did Feshbach and Pease. Parameter searching is much more important in this problem than increasing the number of independent states in the trial wave function.

Whereas these "old-fashioned" potentials all give energies in the general neighborhood of the experimental value, the best and most modern potentials (HJ and Yale) give completely unreasonable results. They barely bind the triton at all, compared to the deuteron binding energy; and He^3 would, with these potentials, be unstable against breakup into deuteron plus proton. In view of the interest attached to these potentials, we spent a great deal of effort on them. The final trial function includes 9 of the 10 states of the triton,[8] and furthermore the principal S state is represented by three exemplars, the principal D state by two. The only state omitted altogether is the fully antisymmetric P state, state 5. The wave function contained 74 nonlinear parameters, of which 55 were varied, many of them repeatedly. In addition, there were 12 amplitudes occurring linearly in the wave function, which were adjusted automatically in each run. The parameter search took more than 100 hours on the IBM 7090.

In view of the very low binding energies, the quantity minimized by the parameter search was not the energy E, but rather a "multiplier" g defined as follows: If all potentials are multiplied by the constant g, then the expectation value of the Hamiltonian over the trial wave function is exactly -8.492 Mev, the experimental value. This assures that all trial wave functions are reasonable triton functions. By contrast, a parameter search to minimize E would have led, probably, to a wave function describing a bound deuteron plus a neutron at a large distance. The energy

Table I. Triton results. The first column contains the potential (see text); the next column contains s, the nondimensional strength parameter of the triplet-even-central force in that potential. $E(\mathrm{H}^3)$ is the triton energy, which should be compared with the experimental value, $E = -8.492$ Mev. R_C is the Coulomb radius,[a] with experimental value 2.26×10^{-13} cm. The last four columns contain the probabilities (in the best wave function) of the principal S state, of state 3 of Derrick and Blatt,[b] of the P states, and of the D states, respectively. A straightforward calculation from the magnetic moments of H^3 and He^3, averaging out the exchange magnetic moment and ignoring all other effects, gives a D-state probability of roughly 4%. Disagreement in this quantity should not be taken seriously.

Potential	s	$E(\mathrm{H}^3)$ (Mev)	R_C (10^{-13} cm)	p_1 (%)	p_3 (%)	p_P (%)	p_D (%)
FP No. 1	1.11	≤-11.2	1.65	96.3	0.68	···	3.0
FP No. 2	1.01	≤-10.1	1.66	96.2	0.52	···	3.3
FP No. 3	0.824	≤- 8.6					
HM	1.155	≤- 8.1	2.12	95.2	1.2	···	3.6
GB	0.621	≤- 5.7	2.50	91.8	0.15	0.036	8.0
HJ	0.403	≤- 2.6	2.47	92.4	0.14	0.028	7.5
Yale	0.435	≤- 2.5	2.50	92.4	0.23	0.029	7.4

[a] See J. M. Blatt and V. F. Weisskopf, Theoretical Nuclear Physics (John Wiley & Sons, Inc., New York, 1952), Chap. II, Sec. 2.

[b] See reference 8.

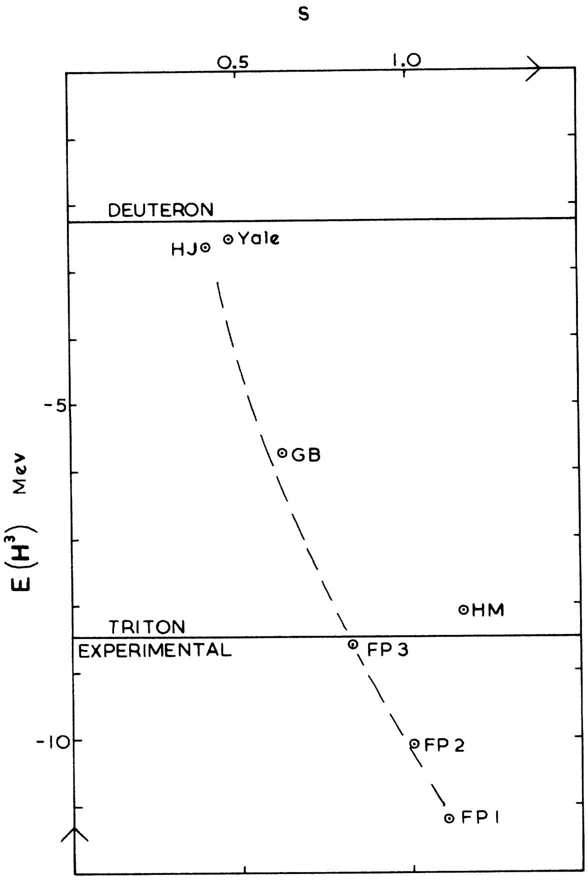

FIG. 1. The triton binding energy $E(H^3)$ is plotted against the strength parameter s of the central-triplet-even potential for each of the potentials listed in Table I. With the exception of the Hu-Massey potential, all the points fall nearly on one smooth curve, shown dashed in the figure. Thus $E(H^3)$ depends very strongly on s. The two horizontal lines indicate the experimental value of $E(H^3) = -8.492$ Mev, and the deuteron energy $E(H^2) = -2.226$ Mev, respectively.

limits quoted in Table I were obtained by setting $g = 1$ again in all matrix elements of the potential energy.

It is instructive to plot our energy limits against the nondimensional strength parameter[9] s of the central-triplet-even potential. Values of s are stated in Table I. The plot, Fig. 1, shows a marked regularity, with E strongly dependent on s. The only exception is the HM point, to which we do not attach much significance, since the range of this potential is so very different, and since (for that reason) we did not do as much parameter searching as for the other potentials.

The trend shown in Fig. 1 is not surprising. The tensor force is much more effective in the deuteron than in the triton; thus a deuteron fit which relies for deuteron binding primarily on the tensor force (such as HJ and Yale) underbinds the triton quite badly.

Figure 1 also suggests strongly that it may not be possible to fit the experimental triton energy with the GB potential. Our trial function for the GB potential was of the same form, with as many parameters, as for the HJ and Yale potentials, and we did a considerable amount of parameter searching, about 50 hours on the IBM 7090.

From the purely phenomenological point of view, the present work does not imply the existence of strong three-body forces. The *np* data are insufficient, at this time, to determine the $T = 0$ forces uniquely. In particular, the strength of the triplet-even-central potential, on which the triton energy depends so sensitively, is very badly determined by present data (although it is reasonably well determined if one imposes meson-theoretical conditions on the fit). Measurements are now in progress[10] which will narrow down this uncertainty appreciably.

The disturbing aspect of these results is the fact that the strong tensor force, and weak central force, in the HJ and Yale potentials are not accidental, but rather are well-nigh inescapable consequences of meson theory. In the one-pion exchange region, the tensor force exceeds the central force by the large factor

$$D(x) = 1 + 3/x + 3/x^2, \qquad (2)$$

where $x = \mu r$. In the two-pion exchange region the calculations disagree, but all of them agree qualitatively in the sense that the tensor force remains strong and dominant.

The factor $D(x)$ is a direct consequence of the Yukawa field equations for the meson field. The S-wave solution for a point source is $\exp(-x)/x$, and the D-wave solution (which is required for a tensor force) is $D(x)\exp(-x)/x$.

It may not be altogether accidental that this large meson-theoretic tensor force also gives uncomfortably (though not impossibly) large D-state probabilities for both the deuteron and the triton; and that it aggravates the trouble in fitting high-energy ^3D phase shifts.[11]

Although meson theory does predict the existence of specifically three-body forces,[12] it is by no means certain that these are large enough, or even of the right sign, to bring the theoretical triton energy down to the experimental value.

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[1] G. Derrick, D. Mustard, and J. M. Blatt, Phys. Rev. Letters 6, 69 (1961).

[2] K. A. Brueckner and J. L. Gammel, Phys. Rev. 109, 1023 (1958).

[3] R. L. Pease, thesis, Massachusetts Institute of Technology, 1950 (unpublished); Phys. Rev. 81, 142L (1951); 88, 945 (1952).

[4] T-M. Hu and H. S. W. Massey, Proc. Roy. Soc. (London) A196, 135 (1949).

[5] T-M. Hu and K-N. Hsu, Phys. Rev. 78, 633 (1950); Proc. Roy. Soc. (London) A204, 476 (1951).

[6] T. Hamada and I. D. Johnston (to be published); we are grateful to Professor Hamada for a prepublication copy of this potential.

[7] K. E. Lassila, M. H. Hull, H. M. Ruppel, F. A. McDonald, and G. Breit, Phys. Rev. (to be published); we are grateful to Professor Breit for a prepublication copy of this potential.

[8] G. H. Derrick and J. M. Blatt, Nuclear Phys. 8, 310 (1958); G. H. Derrick, Nuclear Phys. 18, 303 (1960).

[9] J. M. Blatt and V. F. Weisskopf, Theoretical Nuclear Physics (John Wiley & Sons, Inc., 1952); see Chap. II, Sec. 2.

[10] Richard Wilson, Bull. Am. Phys. Soc. 7, 20 (1962).

[11] T. Hamada, Progr. Theoret. Phys. (Kyoto) 24, 1033 (1960); 25, 247 (1961).

[12] L. Jánossy, Proc. Cambridge Phil. Soc. 35, 616 (1939); H. Primakoff and T. Holstein, Phys. Rev. 55, 1218 (1939); S. D. Drell and K. Huang, Phys. Rev. 91, 1527 (1953).