INTRINSIC STRUCTURE AND LOW-ENERGY $\pi\pi$ SCATTERING IN $K^{\pm}\rightarrow 3\pi$ DECAY

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To a good first approximation the observed invariant amplitude in τ decay is a linear function of the unlike pion energy, with small slope.[1] Khuri and Treiman[2] have calculated the slope on the assumption that it would vanish in the absence of final-state interactions. They use a one-dimensional dispersion representation which they derive heuristically, but which is at least known to hold for the first few orders in perturbation theory.[3] In keeping with their assumption of no intrinsic structure they make one subtraction.

By contrast, it has been pointed out recently that the linear component could equally well be ascribed to the weak (strangeness-changing) coupling either[4] of the $K\pi$ resonance K^*, or[5] of the $\pi\pi$ resonance ρ, both regarded as elementary vector particles in their own right, but unstable already because of their strong interactions.[6] This suggestion is supported by the fact that in order to fit the data, KT need a combination of $T=2$ and $T=0$ $\pi\pi$ scattering lengths somewhat at variance with those derived from analyses of the pion-nucleon system.[7,8] Equivalent conclusions would follow, in the language of perturbation theory, from the presence of terms with derivatives in the weak Lagrangian density, of the general type $(\pi\partial_\mu K)\times(\pi\partial_\mu\pi)$, which leads to a strictly linear structure in the absence of final-state interactions. We would argue that the existence of intrinsic structure is eminently plausible even without reference to the resonances, for a (current) × (current) model of the weak interactions would surely embody terms similar to the one above.[9]

An intrinsic structure, once admitted, makes it very implausible that KT's once-subtracted dispersion representation converges; even if it did, it would be impractical for calculation. In any case, for the iterative solution with scattering length parametrization at least two subtractions are needed, so that both the magnitude and the slope (at some selected point) of the amplitude must be taken from observation. From this point of view, the first components of the spectrum which can yield information about low-energy $\pi\pi$ scattering are the quadratic ones. The situation is exactly analogous to that in perturbation theory with a mixture of nonderivative and derivative weak $K\pi^3$ couplings. There, the constant and the linear terms of the amplitude diverge and must be renormalized independently; the remaining terms, necessarily prefaced by a quadratic factor, are then finite.

The preceding observation is made within the strict letter of the law governing the dispersion-theoretic approach. It does not imply that with the exercise of additional physical insight no information can be extracted from the measured value of the slope. In particular, Bég and DeCelles[5] point out that in the popular pole approximation the slope is indeed determined by the $\pi\pi$ scattering amplitude. Without prejudice to the validity of any bolder approach, we argue merely that the calculated quadratic components have the merit of being less model-dependent, while still on the verge of measurability.

In view of the prospective increase in the number of events analyzed, we have calculated these quadratic components in terms of the $\pi\pi$ scattering lengths, using the twice-subtracted dispersion formalism. The result is, of course, independent of the ancestry ascribed to the linear component.

Except for making an extra subtraction, allowing both S- and P-wave $\pi\pi$ scattering, and including the linear component of the amplitude with the zero-order term for use in the first iteration, our method, notation, and numerical approximations follow KT exactly. Since the procedure is straightforward we present only the results; this note must therefore be read together with KT, to whom we refer for a discussion of the approximations involved. We (i) assume the $\Delta T=\frac{1}{2}$ rule[10]; (ii) include only the two-pion intermediate state; (iii) approximate both S- and P-wave $\pi\pi$ scattering amplitudes by the scattering length parametrization; (iv) confine ourselves to the first iteration; (v) neglect the variation of the dispersion integrals over the physical decay region, and evaluate them only at the symmetric point; (vi) work only to first order in the scattering lengths, so that only the real parts of the dispersion integrals are relevant.

The invariant τ and τ' decay amplitudes are M and M', respectively. Let k_a, k_b, k_c be the four-momenta of the pions, $s_a=(k_b+k_c)^2$, etc., the pion mass being unity and the K mass $M_K=3+\epsilon$. The symmetric point, at which the subtractions are made, is $s_a=s_b=s_c=s_0=(M_K^2+3)/3=5.18$, corresponding to a squared center-of-mass three-momentum of $\vec{k}^2=(s_0-4)/4=k^2$, and energy $\omega=(k^2+1)^{1/2}$. Then

$$M=A+B,\quad M'=C,$$

with

$$A(s_a,s_b,s_c)=A(s_a,s_c,s_b)$$

$$=B(s_b,s_a,s_c)=C(s_c,s_b,s_a).$$

The zeroth approximation consists in writing

$$A=D\{1+\lambda(s_a-s_0)\},\tag{1}$$

with D determined by the absolute decay rate, and λ by its slope. We drop D from here on. Strictly speaking, the slope fixes only Reλ. However, Imλ enters only into the quadratic dependence of the rate, and even if Im$\lambda\approx$Reλ, the entire $|\lambda|^2$ contribution is negligible compared with the rest. (Note also that in the absence of final-state interactions λ would be real by invariance under time reversal, so that Imλ is not expected to be large.) In first iteration, (1) is replaced by

$$A=\{1+\lambda(s_a-s_0)+\tfrac{1}{3}(s_a-s_0)^2[J_0-J_2]$$
$$+[(s_b-s_0)^2+(s_c-s_0)^2]J_2/2$$
$$+[(s_b-s_0)(s_a-s_c)+(s_c-s_0)(s_a-s_b)]H_1/2\},\tag{2}$$

where

$$f_T(s)=\exp(i\delta_T)\sin\delta_T(s),$$

$$F_{T,N}=\pi^{-1}\int_4^\infty ds' f_T(s')(s'-s_0+i\epsilon)^{-N-1},$$

$$J_0=(5F_{0,2}+2\lambda F_{0,1}),\quad J_2=(2F_{2,2}-\lambda F_{2,1}),$$

$$H_1=\lambda F_{1,1}.\tag{3}$$

Above, δ_0, δ_2, δ_1 are the $\pi\pi$ phase shifts for S-wave $T=0$ and $T=2$, and for P-wave $T=1$ scattering, respectively. The integrals F are evaluated in the scattering length parametrization,[2]

$$f_0=a_0k/\omega,\quad f_2=a_2k/\omega,\quad f_1=a_1k^3/\omega^3,\tag{4}$$

in which they are proportional to the scattering length. Thus $F_{2,N}$ is got from $F_{0,N}$ by the replacement $a_0\to a_2$. We find

$$F_{0,1}=\frac{a_0}{4\pi}\left[\frac{\ln(\omega-k)}{k\omega^3}-\frac{1}{\omega^2}\right]=(-0.114)\,a_0;$$

$$F_{0,2}=-\frac{a_0}{64\pi}\left(\frac{1}{\omega^5k^3}\right)[(1+4k^2)\ln(\omega-k)+\omega k(1-2k^2)]$$

$$=1.44\times10^{-2}a_0;$$

$$F_{1,1}=\frac{a_1}{2\pi}\left\{1+\left(\frac{3k}{2\omega^5}\right)\ln[2(\omega-k)]+\frac{k^3\omega-3k\omega^3-k^2}{2\omega^4}\right\}$$

$$=5.10\times10^{-2}a_1.\tag{5}$$

In the K rest frame let us call t the kinetic energy of the unlike pion in units of its maximum value, and let θ be the angle between the momentum of the unlike pion and the relative momentum of the like pions. Define $q=t-\frac{1}{2}$. For comparison with experiment we quote the expressions for the τ mode,

$$R(q)=\int_0^1|M|^2d\cos\theta=1+\Lambda q+C_1q^2,$$

$$R(\cos\theta)=\int_{-1/2}^{1/2}|M|^2dq=1+C_2\cos^2\theta,\tag{6}$$

and the corresponding primed quantities for the τ' mode. They have been separately normalized so that in each case the leading, constant, term is unity. We evaluate the quadratic terms in the nonrelativistic approximation which is good to about 25% and should be adequate to the statistics that can be expected to become available. [If a better approximation is desired, Eqs. (2) and (3) should be taken as the new starting point.] The numbers below are calculated using our results for the $F_{T,N}$ and the rough value 0.185 for λ. We have for the linear terms

$$\Lambda=4\epsilon\lambda,\quad\Lambda'=-\delta\epsilon\lambda,\tag{7}$$

and for the quadratic coefficients

$$C_1=(8\epsilon^2/3)[4J_0+15J_2+6H_1]$$

$$=0.093\,a_0+0.58\,a_2+0.044\,a_1;$$

$$C_1'=(32\,\epsilon^2/3)[J_0-J_2-3H_1]$$

$$=0.093\,a_0-0.16\,a_2-0.088\,a_1;$$

$$C_2 = (4\epsilon^2/3)[2J_0 + J_2 - 3H_1]$$
$$= 0.023\,a_0 + 0.019\,a_2 - 0.011\,a_1;$$
$$C_2' = (4\epsilon^2)[J_2 + H_1] = 0.058\,a_2 + 0.011\,a_1. \qquad (8)$$

Notice that for scattering lengths with a reasonable magnitude the $\cos^2\theta$ dependence is negligible, which agrees with the existing data as far as they go.[1] With an estimated[7] $a_1 \lesssim 0.1$ the effect of the P wave is also negligible.

As regards the scattering length parametrization, it should certainly be adequate for the S-wave[2] integrals, but may be questioned for the P wave, where the ρ resonance impends, though at an energy much above the physical decay region. As a check, we have calculated the real part (which alone is relevant) of the contribution of the resonance to $F_{1,1}$. With the observed width $\Gamma \approx 0.357$ and position $k_\gamma^2 \approx 6.15$, this contribution is dominated by a term

$$\Gamma/4\pi k^2(k_\gamma^2 - k^2) \approx 0.02, \qquad (9)$$

which is negligible in its effects, even though it may be comparable to $F_{1,1}$ evaluated in the scattering length parametrization. Similarly the effects of the K^* are expected to be negligible.

[1]For present data see E. Lomon et al., Ann. Phys. (New York) 13, 359 (1961).
[2]N. N. Khuri and S. B. Treiman, Phys. Rev. 119, 1115 (1960). We shall refer to this paper as KT.
[3]G. Barton and C. Kacser, Nuovo cimento 21, 593 and 988 (1961).
[4]Riazuddin and Fayyazuddin, Phys. Rev. Letters 7, 464 (1961).
[5]M. A. B. Bég and P. DeCelles, Phys. Rev. Letters 8, 46 (1962).
[6]M. Gell-Mann and F. Zachariasen, Phys. Rev. 124, 953 (1961).
[7]H. J. Schnitzer, Phys. Rev. 125, 1059 (1962).
[8]C. Ceolin and R. Stroffolini, CERN Report 2114-Th.212, 1961 (unpublished).
[9]F. Gürsey, Nuovo cimento 16, 230 (1960) and Ann. Phys. (New York) 12, 91 (1961).
[10]S. Weinberg, Phys. Rev. Letters 4, 87, 585(E) (1960).

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BETA-GAMMA PHASE TRANSFORMATION IN SOLID He^3. J. P. Franck [Phys. Rev. Letters 7, 435 (1961)].

The pressure units were quoted as atmospheres; this is wrong. Throughout the Letter "atm" should be replaced by "kg/cm^2."