MAGNETOHYDRODYNAMIC EQUATIONS FOR FINITE LARMOR RADIUS

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In a recent paper Rosenbluth, Krall, and Rostoker[1] discussed the influence of finite Larmor radius on the gravitational instability of a magnetized plasma, finding a stabilizing effect. These authors used the Vlasov equation, but subsequently Lehnert,[2] using the two-fluid magnetohydrodynamic equations, has cast doubt on their result. Lehnert's work appears to show that the stabilizing effect found by Rosenbluth et al.[1] is exactly cancelled by another term from the two-fluid equations, leaving only a residual stabilizing effect which he reported previously.[3]

It will be shown in this Letter that the null result found by Lehnert[2] is a consequence of the fact that he has used only the scalar pressure in the ion equation of motion. We shall find that other terms which are known to exist[4,5] in the ion pressure tensor are actually responsible for the additional effect found by Rosenbluth et al.[1] In fact, the full result obtained by these authors can be derived from the magnetohydrodynamic equations by including known modifications to both the ion pressure tensor and also Ohm's law.[6] The effect found by Lehnert corresponds to modifying Ohm's law only.

The ion pressure tensor must be modified by including certain transport terms. These represent a type of viscosity, independent of any collisions, in which the Larmor radius takes the place of the usual mean free path. The appropriate terms were first deduced from Boltzmann's equation by Chapman and Cowling,[4] and Marshall[5] has supplied a detailed derivation. Because these authors include collisions as well as finite Larmor radius, their calculation of the pressure tensor is very complicated. A simpler method for obtaining the required terms in the collision-free approximation has recently been published by Thompson.[7] The relevant parts of the ion pressure tensor are

$$p_{xx} = p - \rho\nu(\partial V_y/\partial x + \partial V_x/\partial y),$$
$$p_{yy} = p + \rho\nu(\partial V_y/\partial x + \partial V_x/\partial y),$$
$$p_{xy} = p_{yx} = \rho\nu(\partial V_x/\partial x - \partial V_y/\partial y). \tag{1}$$

Here we have assumed that the magnetic field $\vec{B}$ is in the z direction and essentially uniform, while the plasma velocity is in the xy plane. p is the ion perpendicular pressure and ρ is the density. The parameter ν has the dimensions (but not the exact physical significance) of a kinematic viscosity, and is defined by $\nu = a^2\Omega/4$, where a is the ion Larmor radius and Ω is the ion gyrofrequency. The transport terms in the electron pressure tensor can be neglected.

As an illustration of the effect of this modification of the pressure tensor we shall derive the dispersion relation contained in Eq. (2.13) of reference 1. For this purpose it is convenient to use the equivalent one-fluid equations. Suppose that there is a uniform gravitational field $\vec{g}$ in the x direction, and a uniform density gradient $\partial\rho_0/\partial x = -\eta\rho_0$. If p now denotes the total perpendicular pressure, the plasma equation of motion is

$$\frac{\rho D\vec{V}}{Dt} = -\vec{\nabla}\left(p + \frac{B^2}{8\pi}\right) + \rho\vec{g} + \nu\vec{\lambda}, \tag{2}$$

where

$$\lambda_x = \frac{\partial}{\partial x}\left[\rho\left(\frac{\partial V_y}{\partial x} + \frac{\partial V_x}{\partial y}\right)\right] - \frac{\partial}{\partial y}\left[\rho\left(\frac{\partial V_x}{\partial x} - \frac{\partial V_y}{\partial y}\right)\right],$$

$$\lambda_y = -\frac{\partial}{\partial y}\left[\rho\left(\frac{\partial V_y}{\partial x} + \frac{\partial V_x}{\partial y}\right)\right] - \frac{\partial}{\partial x}\left[\rho\left(\frac{\partial V_x}{\partial x} - \frac{\partial V_y}{\partial y}\right)\right]. \tag{3}$$

The continuity equation is

$$\partial\rho/\partial t + \vec{\nabla}\cdot(\rho\vec{V}) = 0. \tag{4}$$

If we retain the simple relation

$$\vec{E} + \vec{V}\times\vec{B} = 0, \tag{5}$$

then by taking the curl of (5) and using the low-β approximations

$$\vec{\nabla}\times\vec{E} = 0, \quad \vec{\nabla}\times(\vec{V}\times\vec{B}) = -\vec{B}(\vec{\nabla}\cdot\vec{V})$$

we find

$$\vec{\nabla}\cdot\vec{V} = 0. \tag{6}$$

Equations (2), (4), and (6) can now be solved by taking the curl of (2), linearizing, and looking for solutions of the form $\exp(i\omega t + iky)$, since the x dependence is treated as weak in reference 1. This leads to the dispersion relation

$$\omega^2 + 2\nu\eta k\omega + g\eta = 0, \tag{7}$$

which represents the difference between the result of Rosenbluth et al.[1] and that of Lehnert.[2]

To obtain the full result we use the generalized Ohm's law[6]:

$$\vec{E} + \vec{V} \times \vec{B} + (c/en)\vec{\nabla} p_e - (c/en)\vec{j} \times \vec{B} = 0, \quad (8)$$

where n is the number density, p_e is the electron pressure, and $\vec{j}$ is the current density. Electron inertia and finite resistivity have been neglected in (8). The last two terms of (8) can be transformed with the equation of motion to give

$$\vec{E} + \vec{V} \times \vec{B} - \frac{c}{en}\vec{\nabla} p_i - \frac{Mc}{e}\frac{D\vec{V}}{Dt} + \frac{Mc}{e}\vec{g} + \frac{\nu c}{ne}\vec{\lambda} = 0, \quad (9)$$

where p_i is the scalar ion pressure and M is the ion mass. Taking the curl of (9) as before, we find that (6) is replaced by

$$\vec{b}(\vec{\nabla} \cdot \vec{V}) + \frac{1}{\Omega}\vec{\nabla} \times \left(\frac{D\vec{V}}{Dt}\right) - \frac{\nu}{\Omega}\vec{\nabla} \times \left(\frac{1}{\rho}\vec{\lambda}\right) = 0, \quad (10)$$

where $\vec{b}$ is a unit vector in the direction of the magnetic field. [The third term of (9) does not contribute to (10) if we assume with Lehnert[2] that temperature variations can be neglected.] Solving (2), (4), and (10) as before by taking the curl of (2) and linearizing, we find the dispersion relation. This must now be cubic in ω, but takes the form $F_0(\omega) + \delta = 0$, where $F_0(\omega)$ is a quadratic expression and δ is a small additional contribution whose effect we can show to be negligible.[8] In fact δ displaces each root ω_0 of the relation $F_0(\omega) = 0$ by an amount $\mu\omega_0$, where μ contains factors $(ak)^2$, $(a\eta)^2$, and ω/Ω which are treated as small in reference 1. The final result is therefore

$$F_0(\omega) \equiv \omega^2 + (2\nu\eta k + gk/\Omega)\omega + g\eta = 0. \quad (11)$$

Equation (11) gives exactly the same stability condition as (2.14) of Rosenbluth et al.,[1] when account is taken of the difference between their ϵ_1' and our η, namely

$$\epsilon_1' = \eta + 2g/(a^2\Omega^2) = \eta + g/(2\nu\Omega). \quad (12)$$

There is an apparent difference of sign between the coefficient of ω in Eq. (11) and that in Eq. (2.13) of reference 1. This difference arises because Rosenbluth et al.[1] use a coordinate system in which the unperturbed electric field is zero, while we assume that the bulk velocity of the plasma is zero. The relative velocity V_D of the two systems is determined by the equilibrium equation for the ions, i.e., by

$$n(e/c)E_x - \partial p_i/\partial x + \rho g = 0, \quad (13)$$

from which

$$V_D = E/B = 2\nu\eta + g/\Omega. \quad (14)$$

When the corresponding Doppler shift in ω is allowed for, the two dispersion relations become identical. Lehnert's result[2] is obtained by setting $\nu = 0$ in (11).

We have therefore shown that the finite Larmor radius stabilization demonstrated by Rosenbluth et al.[1] can be obtained from the magnetohydrodynamic equations, by making prescribed modifications to the ion pressure tensor[4] and to Ohm's law.[6] It is not essential to use Vlasov's equation for this type of problem, and the more elementary derivation given here may help to make the physics clearer.

The only significant contribution from the $\vec{\lambda}$ term in (10) is a term $g\eta(a^2k^2/4)$ in δ, which can be ignored because it is equivalent to a negligible change in g. Therefore (10) can be written more simply as

$$\vec{b}(\vec{\nabla} \cdot \vec{V}) + (1/\Omega)\vec{\nabla} \times (D\vec{V}/Dt) = 0. \quad (15)$$

If temperature variations had been allowed for in (9), their effect would have been equally small. In general, cross terms between the two effects introduced by modifying Ohm's law and the ion pressure tensor can be neglected because these modifications cannot simultaneously be important, since

$$(2\nu\eta k)(gk/\Omega) = g\eta(a^2k^2/2). \quad (16)$$

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[1]M. N. Rosenbluth, N. A. Krall, and N. Rostoker, Conference on Plasma Physics and Controlled Nuclear Fusion Research, Salzburg, September, 1961 (unpublished), Paper CN-10/170.

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[3]B. Lehnert, Phys. Fluids 4, 525 (1961); 4, 847 (1961); reference 1, Paper CN-10/3.

[4]S. Chapman and T. G. Cowling, The Mathematical Theory of Nonuniform Gases (Cambridge University Press, New York, 1952), 2nd ed., p. 338. The 1939 edition has a sign error in these terms.

[5]W. Marshall, Harwell Report AERE T/R 2419, 1958 (unpublished).

[6]L. Spitzer, Jr., Physics of Fully Ionized Gases (Interscience Publishers, Inc., New York, 1956), Eq. (2.12), p. 21.

[7]W. B. Thompson, Reports on Progress in Physics (The Physical Society, London, 1961), Vol. 24, p. 363.

[8]The third root has a high frequency $\omega \simeq k\Omega/\eta$ and is not relevant to the present discussion.