sion of $P_\alpha(z)$ for large z. The more exact expressions will be considered in another communication in connection with the angular distributions in the diffraction peak.

[12]In our table we have included only the known Regge poles whose trajectories lie high enough for them to show up as particles. These appear to be sufficient for a description of the cross sections in the region of several Bev. At low energies more Regge poles may be expected to contribute as well.

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[16]We have assumed here that η has spin 1. There seems to be some uncertainty now about this assignment. If the spin turns out to be 0, the effect of η on high-energy cross sections may not be very large.

P-WAVE PION-PION RESONANCE

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In a recent paper[1] solutions of the coupled S- and P-wave amplitudes in pion-pion scattering were obtained for the range in the coupling constant, $0.20 \leq |\lambda| \leq 0.45$, in which the P wave exhibits a resonance, and the S-wave scattering length for the $I=0$ isotopic spin state is greater than unity.[2] We present here the results for the solution corresponding to $\lambda = -0.1$, which satisfies the requirements of analyticity, unitarity, and crossing symmetry at low energies.[3] This solution possesses the features of the recent experimental results on the P-wave resonance in the two-pion system.[4-7]

In obtaining the solutions for smaller $|\lambda|$, the same methods were used as in reference 1. In particular the values of the two P-wave constants ξ_1 and a_1 (quoted in reference 1) were determined by the first and second derivative crossing conditions at $\nu = -\frac{2}{3}$.[8] The system of equations for low-energy pion-pion scattering solved in reference 1 was reprogrammed for numerical solution on the 7090 computer, and the results presented in reference 1 were reproduced. The greater speed of the 7090 computing machine provided the possibility of increasing the number of cycles necessary to reach convergence for smaller $|\lambda|$, and thirty cycles determined the phase shifts to a sufficient accuracy. As explained in reference 1, for the value $\lambda = -0.1$, the calculations are carried out for a number of ξ_1, a_1, the final values being those which satisfy the crossing conditions. As was found previously, the approximate determination of ξ_1, a_1 by the crossing conditions is nontrivial and the width Γ of the P-wave resonance is sensitive to the value of ξ_1 obtained from these conditions.

For the value $\lambda = -0.1$ the calculated values of a_1, ξ_1, the S-wave scattering lengths, and the resonance parameters are shown in Table I.[9] The width Γ is the total width at half maximum, and is related to the reduced width γ by $\Gamma = [\nu_R^3/(\nu_R+1)]^{1/2}\gamma$. The resonance position quoted in Table I corresponds in units of the pion mass to 700 Mev, while the total resonance width at half maximum is of the order 63 Mev.[10] The scattering length α_0 is less than unity, and therefore will not contradict the experiments on pion production in pion-nucleon collisions in processes of the type $\pi + N \rightarrow \pi + \pi + N$.[11-14]

Table I. Calculated values of a_1^1, ξ_1, resonance parameters ν_R, Γ, and S-wave scattering lengths α_0, α_2 for $\lambda = -0.1$.

a_1^1	ξ_1	ν_R	Γ	α_0	α_2
-0.0050	-199.5	5.25	0.2	0.67	0.2

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Note added in proof. Ball and Wong[15] have recently published some criticisms of the solutions given in reference 1, and we should like to comment on these. In the first place there is no "mathematical error" in our iterative procedure. They point out that for $\nu < -15$, $\mathrm{Im}A_l(\nu)$ obtained from our solution becomes different from that given by the approximate crossing equations, a fact already noted in reference 1, p. 510. If we demand that for $\nu < -15$ the left cut from our solution equals the discontinuity obtained from the approximate crossing equation, then the "potential" $\mathrm{Im}[A_l^{-1}(\nu)]$ in our inverse amplitude equations blows up as $\nu \to -\infty$, and must be rejected as physically unacceptable, because it violates unitarity. It is well known that for large negative ν, $\mathrm{Im}A_l(\nu)$ from the crossing equation cannot be correct as the Legendre polynomial expansion employed fails to converge for $\nu < -9$. When the double spectral function is neglected in the calculation of the left cut, a truncation of the expansion beyond the P wave leads to an asymptotic behavior of $\mathrm{Im}A_1(\nu)$ as $|\nu| \to \infty$ that is inconsistent with unitarity (see reference 3 for a more detailed discussion of this point). It seems reasonable to us to construct a solution with asymptotic properties that do not conflict with unitarity, which requires that $\mathrm{Im}A_1(\nu) \to 0$ on the left cut (or perhaps oscillates) and that $\mathrm{Im}[A_1^{-1}(\nu)] \to \mathrm{const}$ as $\nu \to -\infty$. For this physically acceptable solution, $\mathrm{Im}[A_l^{-1}(\nu)]$ is small and very insensitive to $\mathrm{Im}A_l^{-1}(\nu)$ for large $|\nu|$. Cutting off $\mathrm{Im}[A_l^{-1}(\nu)]$ at some large value $|\nu_c|$ on the left cut amounts therefore to neglecting some small constant contribution and the low-energy phase shifts are insensitive to this constant contribution for $\nu < -15$. Our approximation method provides a continuation of $\mathrm{Im}[A_l^{-1}(\nu)]$ for large $|\nu|$ that is consistent with unitarity, and at the same time generates a solution that satisfies the valid crossing equation on the nearby left cut for $-15 \lesssim \nu < -1$.

A further remark of Ball and Wong is that the discontinuity in the P wave on the left has a resonant-type behavior near the zero of $\mathrm{Re}[A_1^{-1}(\nu)]$ at some negative point $-\nu_1$ on the left cut. A study of $\mathrm{Re}[A_1^{-1}(\nu)]$ on the left for large $|\nu|$ shows that this function has a zero for $\nu_1 \approx \exp[\pi c/(1-K)]$, where K enters into the asymptotic behavior of the left-cut contribution to $\mathrm{Re}[A_1^{-1}(\nu)]$, through

$$N_1(\nu) = -\frac{(\nu-\nu_0)}{\pi}\int_1^{\infty}\frac{d\nu'\,\mathrm{Im}[A_1^{-1}(-\nu')]}{(\nu'+\nu)(\nu'+\nu_0)} \sim -K\ln|\nu|, \quad (1)$$

and $C = 1/a_1 - \xi_1$ ($1/a_1$ and ξ_1 are the two constants which appear in the determination of the P wave in reference 1). The location of the zero is clearly sensitive to the magnitude and sign of K. We have

$$\mathrm{Im}[A_1^{-1}(\nu)] = -\mathrm{Im}A_1(\nu)/\{[\mathrm{Re}A_1(\nu)]^2 + [\mathrm{Im}A_1(\nu)]^2\}, \quad (2)$$

and a zero in $\mathrm{Re}[A_1^{-1}(\nu)]$ causes $\mathrm{Re}A_1(\nu)$ to vanish at the same point, such that

$$\mathrm{Im}[A_1^{-1}(\nu)] \sim -1/\mathrm{Im}A_1(\nu), \quad (3)$$

and $\mathrm{Im}A_1(\nu)$ has a resonant behavior in the neighborhood of ν_1, due to a displaced pole in the complex plane. In order to study the consequences of this pole, the calculations in reference 1 were repeated for $\lambda = -0.35$ with a cutoff $|\nu_c| = 5\times10^7$. A zero in $\mathrm{Re}A_1(\nu)$ was found at $\nu_1 \sim 10^4$ and the signs of $\mathrm{Re}[\nu/A_1(\nu)]$ and $\mathrm{Im}[\nu/A_1(\nu)]$ near ν_1 show that the complex pole occurs on the unphysical Riemann sheet. The P-wave phase shift obtained from this calculation was compared with the corresponding phase shift calculated with $\lambda = -0.35$, but with $\mathrm{Re}A_1(\nu)$ and $\mathrm{Im}A_1(\nu)$ set equal to zero for $\nu < -15$. The two P-wave phase shifts differed by less than 2 %. This means that the phase shifts calculated from $A_l(\nu)$ including the distant P-wave pole in the unphysical sheet generate the same low-energy resonance as the amplitude from which not only this pole, but the whole of the discontinuity in the segment $-10^7 \lesssim \nu < -15$ has been subtracted.

The fact that this far-away complex pole in the unphysical sheet does not influence the low-energy P-wave resonance can be understood by the following examples. At the point where $\mathrm{Re}[A_1^{-1}(\nu)] = 0$, $\mathrm{Im}A_1(\nu)$ from crossing is equal to that from our solution, and is equal to $\{\mathrm{Im}[A_1^{-1}(\nu)]\}^{-1}$. The region in the neighborhood of the zero ν_1 contributes to the integral $N_1(\nu)$, in (1), less than

$$\left(\frac{\nu}{\pi}\right)\int_{10^3}^{10^5}\frac{d\omega[5.8]^{-1}}{(\omega+\nu)\omega}, \quad (\nu' = -\omega) \quad (4)$$

which for $\nu = 4$, for example, is of the order 10^{-3}, and is small compared with the "nearby" left-cut contribution of the order -2 to -3 for $\nu = 4$. If we represent $\mathrm{Im}A_1(\nu)$, in the neighborhood of the zero in $\mathrm{Re}[A_1^{-1}(\nu)]$, by a δ-function discontinuity of the form

$$\mathrm{Im}A_1(\nu) = \pi R\delta(\omega - \nu_1), \quad (5)$$

where R measures the strength of the pole, then the pole contribution in the once-subtracted dispersion relation for $\mathrm{Re}A_1(\nu)$ has the form

$$P(\omega)=\frac{(\nu-\nu_0)R}{(\nu_1+\omega)(\nu_1+\nu_0)}.\quad (\nu_0=-\tfrac{2}{3}) \qquad (6)$$

We must consider the "renormalized" pole contribution, because the once-subtracted dispersion relation for $\mathrm{Re}A_1(\nu)$ forms the basis of our theory. It is easily shown by calculating R that $P(\nu)\to 0$ as $\nu_1\to\infty$ and that for $\nu_1\sim 10^4$, $P(\nu)$ is unimportant in the dispersion relation. In particular, $P(\nu)$ has no influence on the determination of the position and width of the low-energy resonance in the P wave.

The resonance position and width are determined to a large extent by the subtraction constants a_1, ξ_1, which are themselves determined by the S waves through the derivative conditions at $\nu=-\frac{2}{3}$. The S waves are only weakly dependent on the P waves. This conclusion has been reached independently by Jacob, Mahoux, and Omnès.[16] The strong dependence on the far-away left cut found by Ball and Wong[15] is presumably due to their use of the N/D equations, which diverge rapidly under iteration and are therefore strongly cutoff dependent. Had they varied their cutoffs a different conclusion might well have been reached.

To conclude, we emphasize that until double spectral functions are included in the calculation, it is literally impossible to determine $\mathrm{Im}A_l(\nu)$ on the far-away left cut. Naturally, the contribution from the distant left cut cannot be entirely arbitrary and we have deliberately chosen to satisfy the boundary condition $\mathrm{Im}[A_l^{-1}(\nu)]\to\mathrm{const}$ as $\nu\to-\infty$, which is consistent with unitarity. The only other behavior which could be envisaged for this function is an infinite oscillation as $\nu\to-\infty$. With this in mind, and also keeping in mind the initial assumption of small D waves and small inelastic effects on the right cut, we believe that our solution represents one form of a consistent, one-parameter, low-energy theory.

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