Stable Infinite Variance Fluctuations in Randomly Amplified Langevin Systems

Hideki Takayasu
Sony Computer Science Laboratory, Takanawa Muse Building, 3-14-13 Higashi-Gotanda, Shinagawa-ku, Tokyo 151, Japan

Aki-Hiro Sato
Graduate School of Information Sciences, Tohoku University, Sendai 980-77, Japan

Misako Takayasu
"Research for the Future" Project, Faculty of Sciences and Technology, Keio University, Shin-Kawasaki-Mitsui Building West 3F, 890-12 Kashimada Saiwai-ku, Kawasaki-shi, Japan, 221

(Received 26 December 1996; revised manuscript received 21 March 1997)

A general discrete stochastic process involving random amplification with additive external noise is analyzed theoretically and numerically. Necessary and sufficient conditions to realize steady power law fluctuations with divergent variance are clarified. The power law exponent is determined by a statistical property of amplification independent of the external noise. By introducing a nonlinear effect a stretched exponential decay appears in the power law. [S0031-9007(97)03737-X]

PACS numbers: 05.40.+j, 02.50.–r, 05.70.Ln, 64.60.Lx

Power law distributions have been found in diverse fields of science and the subjects of physicists' research are growing wider. Quantitative analyses are now in progress on a variety of topics showing power law distributions, for example, fish school sizes [1], frequency of jams in Internet traffic [2], and even stock market price changes [3]. In view of statistical physics the most important task is to elucidate general physical mechanisms underlying these power law behaviors.

A recently proposed concept of self-organized criticality is based on the idea that open systems showing power law distributions may have a mechanism of controlling inherent parameters automatically to be at the critical point of second order phase transition [4]. This idea has been confirmed theoretically for a sandpile model of avalanches that the critical point is stable in the renormalized macroscopic limit [5], and its applications are increasing rapidly.

Another general mechanism of producing power laws has been found in the study of stochastic processes involving multiplicative noises [6]. A typical equation of multiplicative process is given by a linear Langevin equation with a randomly changing coefficient. The effect of such a random coefficient has been intensively analyzed relating to the study of nonlinear dynamical systems because statistical properties of some nonlinear systems can be approximated by such stochastic equations [7,8]. It is intuitively obvious that multiplicative noises drastically enhance the additive random force in the Langevin equation and we have much larger fluctuations than in the case of constant coefficient. Numerical study and theoretical approaches strongly indicate the existence of a statistically steady state in which temporal fluctuations follow a power law distribution for a wide range of parameters in the random coefficient.

Theoretical analysis of the Langevin equation with a random coefficient is generally very difficult, because the master equation cannot be reduced to a solvable Fokker-Plank equation due to large fluctuations except in very special cases [9]. By this approach a sophisticated approximate theory has clarified that the Langevin equation with a random coefficient follows a steady distribution whose tails decay either following a stretched exponential form or a power law [10].

There is a powerful theoretical method for distributions with large fluctuations, the characteristic functions. The characteristic function is a Fourier transform of the probability density and the power law tails for an infinite variance distribution can be represented by a singularity at the origin of the corresponding characteristic function. The mathematical theory of stable distributions, which have power tails, is based on the characteristic functions [11], and some physical systems showing power law distributions have been solved rigorously by using characteristic function techniques [12].

In this paper we focus on temporal fluctuations having infinite variances. We introduce a discrete time version of the Langevin equation with a random coefficient and solve the steady state solution by introducing the characteristic function. We show rigorously that the tails of steady state probability density follow a power law in a very wide range of parameters. The necessary and sufficient conditions to realize the power laws with divergent variance is clarified; also the uniqueness and stability of the power law solution is proved theoretically. The exponent of the power law is not universal but changes continuously depending on the statistics of the coefficient. An exact formula is found for the exponent which clearly shows that the exponent is independent of the statistics of additive random force although the random force is necessary to realize the steady state. We confirm these results also by numerical simulations. In the final part of the paper we discuss briefly a possible direct application to the distribution of stock

 0031-9007/97/79(6)/966(4)$10.00

market price changes and rapid decays of distribution tails due to a nonlinear effect.

The model equation is given by the following simple discrete version of the linear Langevin equation:

$$x(t+1) = b(t)x(t) + f(t), \tag{1}$$

where $f(t)$ represents a random additive noise as usual, and $b(t)$ is a non-negative stochastic coefficient which means dissipation for $b(t) < 1$ and magnification for $b(t) > 1$. The case of magnification never occurs in a stable thermal equilibrium because it corresponds to "negative viscosity" in the continuum Langevin equation. However, magnification of fluctuations occurs in unstable systems in general; therefore, we believe Eq. (1) is a very basic starting point for general phenomena. In the following discussion, for simplicity, we assume that $b(t)$ and $f(t)$ are independent white noises having stationary statistics and $f(t)$ is symmetric.

Taking the average over the square of Eq. (1) we have the following equation for the second order moment:

$$\langle x^2(t+1)\rangle = \langle b^2\rangle\langle x^2(t)\rangle + \langle f^2\rangle, \tag{2}$$

where $\langle\cdots\rangle$ denotes an average over realizations.

As $\langle b^2\rangle$ and $\langle f^2\rangle$ are constants we can readily solve Eq. (2). For $\langle b^2\rangle < 1$ there is a stationary solution,

$$\langle x^2\rangle = \frac{\langle f^2\rangle}{1-\langle b^2\rangle}. \tag{3}$$

In the case of thermal equilibrium the principle of equipartition of energy requires that $\langle x^2\rangle$ is proportional to the temperature, so that $\langle f^2\rangle$ and $\langle b^2\rangle$ cannot be independent as known by the name of fluctuation-dissipation theorem [13]. For $\langle b^2\rangle > 1$ there is no stationary solution for $\langle x(t)^2\rangle$ and it diverges as $t \to \infty$. All higher moments diverge in the same way and it is common sense that such divergence means that the system is not statistically stationary. However, this common sense turns out to be wrong, as we prove in the following discussion. We have statistically steady fluctuation with infinite variance in the limit of $t \to \infty$.

Let the distribution functions of $b(t)$ and $f(t)$ be $W(b)$ and $U(f)$, respectively, which are assumed to be independent of time. The statistics of $x(t)$ is estimated theoretically by introducing the characteristic function, $Z(\rho,t)$, which is the Fourier transform of its probability density, $p(x,t)$:

$$Z(\rho,t) = \langle e^{i\rho x(t)}\rangle = \int_{-\infty}^{\infty} e^{i\rho x} p(x,t)\,dx. \tag{4}$$

Fourier transform of Eq. (1) gives the following basic equation for the characteristic function:

$$\begin{aligned} Z(\rho,t+1) &= \langle e^{i\rho b(t)x(t)}\rangle\langle e^{i\rho f(t)}\rangle \\ &= \int_0^{\infty} W(b)Z(b\rho,t)\,db\;\Phi(\rho), \end{aligned} \tag{5}$$

where $\Phi(\rho)$ is the characteristic function for the additive noise, $f(t)$. By assuming Taylor expansion around $\rho = 0$ Eq. (5) derives a set of equations for moments including Eq. (2) for the lowest order. When the variance diverges $Z(\rho,t)$ have singularity at $\rho = 0$ in the limit of $t \to \infty$ and Taylor expansion cannot be applied for the steady solution. In such a case the following fractional power term can be assumed for the lowest order term because the characteristic function is generally a continuous function [11],

$$Z(\rho,\infty) = 1 - \text{const} \times |\rho|^{\beta} + \ldots, \qquad 0 < \beta < 2, \tag{6}$$

which is equivalent to the assumption of power law tails in the probability distribution:

$$P(\geq|x|) \propto x^{-\beta}, \tag{7}$$

where $P(\geq|x|)$ represents the cumulative distribution defined as

$$P(\geq|x|) = \int_{-\infty}^{-|x|} p(x')\,dx' + \int_{|x|}^{\infty} p(x')\,dx'. \tag{8}$$

Introducing the steady solution's functional form, Eq. (6), into Eq. (5), we have the following consistency condition for the lowest order of ρ in the case that the additive noise's variance is finite and $\Phi(\rho)$ is expanded in integer powers of ρ.

$$\langle b^{\beta}\rangle = 1. \tag{9}$$

For a given distribution of $W(b)$ Eq. (9) can be viewed as the equation determining the value of the singularity exponent, β.

Noting that the function $G(\beta) \equiv \langle b^{\beta}\rangle$ satisfies $G(0) = 1$ and $G''(\beta) > 0$ for $\beta > 0$, we have the following necessary conditions in order to have β in the range of $(0,2)$:

$$\lim_{\beta\to+0} G'(\beta) = \langle \ln b\rangle < 0, \tag{10}$$

$$G(2) = \langle b^2\rangle > 1. \tag{11}$$

The latter condition (11) is obviously the condition for the divergence of variance, $\langle x^2\rangle = \infty$. The former condition (10) corresponds to the requirement of stationarity, namely, if this inequality does not hold the magnification rate is so strong that we do not have a statistically steady state.

We can prove the uniqueness and stability of the steady solution (6) in the following way. Assuming the existence of a steady solution of Eq. (5) the deviation from the steady solution, $\bar{Z}(\rho,t) \equiv Z(\rho,t) - Z(\rho)$, satisfies the same equation with a different boundary condition, $\bar{Z}(0,t) = 0$. By taking absolute values of the equation we have an inequality:

$$|\bar{Z}(\rho,t+1)| \leq \max\{|\bar{Z}(\rho,t)|\}\,|\Phi(\rho)|, \tag{12}$$

where $\max\{\cdots\}$ shows the maximum value. Therefore, in the case $|\Phi(\rho)| < 1$ for $\rho \neq 0$, which is satisfied whenever the external noise is continuously distributed,

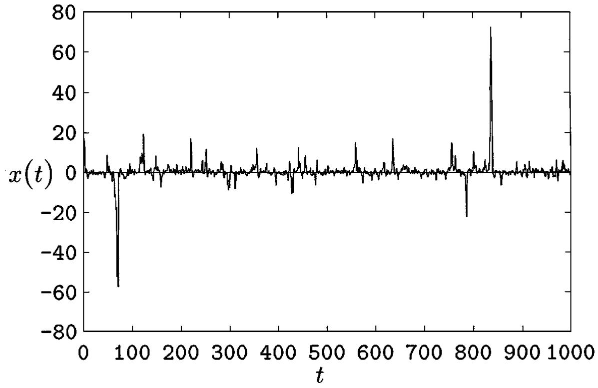

FIG. 1. An example of temporal fluctuations for $c = 0.3$.

the distribution of x converges quickly to the steady solution, (6), even starting from any initial distribution of $\{x(0)\}$. Namely, the conditions (10) and (11) are necessary and sufficient conditions for the power law with infinite variance.

Numerical simulation of Eq. (1) can be done easily. In order to specify the statistics we set the following distributions for b and f, respectively:

$$W(b) = \frac{1}{c}\sum_{k=0}^{\infty}(1 - e^{-\gamma})e^{-\gamma k}\delta\left(\frac{b}{c} - k\right), \tag{13}$$

$$U(f) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-f^2/2\sigma^2}. \tag{14}$$

Here, the variable b takes a discrete value in $\{0, c, 2c, 3c, \ldots\}$ following the Poisson distribution, and f takes a continuous value following the symmetric Gaussian. The second order moment of b is given as $\langle b^2\rangle = c^2e^{-\gamma}(1 + e^{-\gamma})/(1 - e^{-\gamma})^2$, and the distribution $W(b)$ is controlled by a non-negative parameter c. In our numerical simulations the maximum time steps are typically 5×10^7 and we observe the distribution of $\{x\}$ for time steps after 1000. Figure 1 shows a typical example of temporal fluctuations for $\langle b^2\rangle > 1$ with $\gamma = 0.32$, and $\sigma = 0.86$, which we chose for convenience of numerical calculations. As shown in Fig. 2 we can find a clear power law tail in the cumulative distribution of $x(t)$, $P(\geq|x|)$.

By repeating numerical calculations several times for each parameter we have confirmed that the power law exponent is independent of initial conditions, seeds of random number generator, and the functional forms of $U(f)$, as expected.

For different values of c the power law exponents are estimated numerically as shown in Fig. 3. In the case that distribution of b is given by Eq. (13) we can derive an analytic relation between β and c from Eq. (9) [14]:

$$c^{\beta}(1 - e^{-\gamma})\Gamma(\beta + 1)/\gamma^{\beta+1} = 1\,, \tag{15}$$

where $\Gamma(\beta)$ is the gamma function. We can confirm from Fig. 3 that Eq. (15) fits the numerical estimation quite nicely.

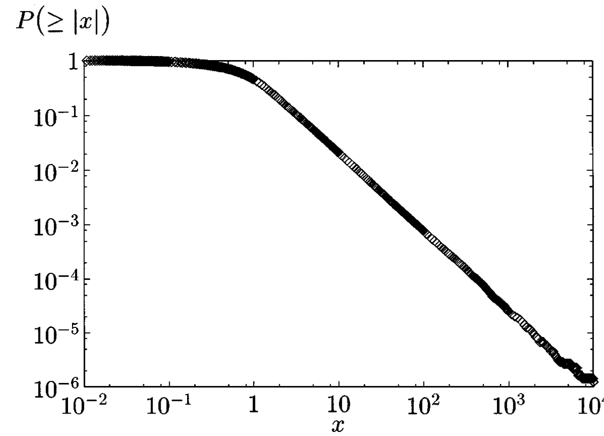

FIG. 2. Log-log plot of the cumulative distribution of x for $c = 0.3$.

It should be remarked that the theoretical estimate of Eq. (15) shows nice fits even out of the range of applicability, $\beta > 2$. The reason for this lucky coincidence is not clear but it is easy to tell that power law distribution tails are a generic property of Eq. (1). Actually, if the probability measure of $b(t) > 1$ is nonzero there exists a real number n_c and $\langle x(t)^n\rangle$ is divergent for $n \geq n_c$. This singularity implies that the distribution of $x(t)$ in the steady state has a power tail of Eq. (7) with $\beta = n_c$ [10]. On the contrary in the case of no magnification we have an analytic solution for $Z(\rho)$ and the distribution tails decay faster than any power.

Since the Langevin equation or its discrete version is one of the most basic stochastic equations not only in physics, potential applicability of our result is expected to be very wide. A direct application can be found in the cross-disciplinary field between statistical physics and economics. Stanley and his co-workers recently discovered nontrivial scaling relations in economic activities such as stock market prices [15]. It is pointed out that the distribution of averaged stock market price changes are well

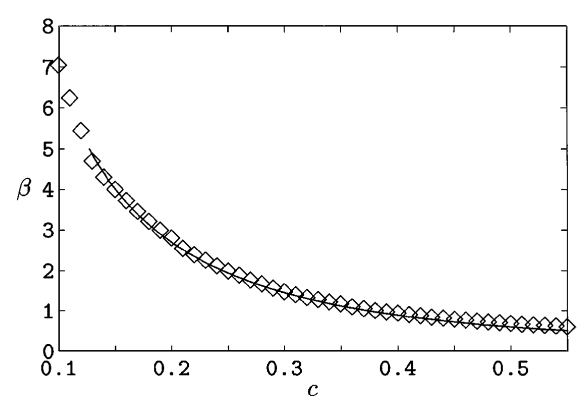

FIG. 3. The power law exponent β vs the amplification parameter c. Squares represent numerically estimated values and the curve gives the theoretical relation, Eq. (15).

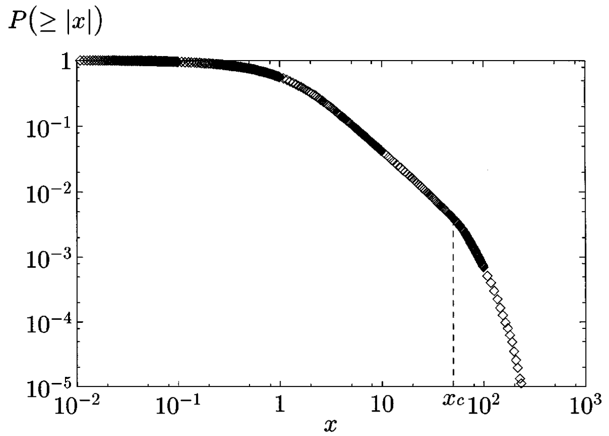

FIG. 4. Log-log plot of the cumulative distribution with the rapid decay. The threshold value is $x_c = 50.0$ with $c = 0.3$.

approximated by a symmetric power law distribution with an exponent about $\beta = 1.4$. The present authors (H. T. and A.-H. S.) have developed mathematical models of stock market prices and showed that price changes of a market can be approximated by Eq. (1) with the distribution of $b(t)$ given by Eq. (13) [14]. Intuitively a stock price change in a unit time is either magnified or damped randomly with an additive external noise. Our general result summarized in Fig. 3 immediately indicates that the best parameter for describing the long tail of real economic data can easily be estimated.

In real systems there is no rigorous power law distributions, but power tails are normally accompanied with rapid decays for very large values. Our model equation, Eq. (1), can easily be modified to manage this deviation from the power law. In Eq. (1) we assume that $b(t)$ is independent of $x(t)$; however, a system size limitation, for example, may introduce a correlation between $b(t)$ and $x(t)$. As a simplest modification we assume that $b(t) < 1$ for $|x(t)| > x_c$, where x_c is a given threshold value. In Fig. 4 a steady state distribution with this modification is shown. We can find a power law distribution with a rapid decay around x_c which can be approximated by a stretched exponential form.

Summarizing the results we have clarified necessary and sufficient conditions for a quantity described by Eq. (1) to follow a power law distribution with divergent moment. This is a new general route to power law fluctuations and it is now obvious that the divergence of variance is the most essential key ingredient for power law distributions.

The authors acknowledge H. E. Stanley, Y. Kuramoto, H. Nakao, and H. Hayakawa for valuable discussions. This research is partially supported by the Japan Society for the Promotion of Science, "Research for the Future" Program No. JSPS-RFTF96P00503.

[1] E. Bonabeau and L. Dagorn, Phys. Rev. E **51**, R5220 (1995).
[2] M. Takayasu, H. Takayasu, and T. Sato, Physica (Amsterdam) **233A**, 824 (1996).
[3] P. Bak, K. Chen, J. A. Scheinkman, and M. Woodford, Ric. Economichi **47**, 3 (1993); M. H. R. Stanley *et al.*, Nature (London) **379**, 804 (1996).
[4] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. A **38**, 364 (1988).
[5] L. Pietronero, A. Vespignani, and S. Zapperi, Phys. Rev. Lett. **72**, 1690 (1994).
[6] M. Levy and S. Solomon, Int. J. Mod. Phys. C **7**, 595 (1996).
[7] S. C. Venkataramani *et al.*, Physica (Amsterdam) **96D**, 66 (1996); A. Čenys and H. Lustfeld, J. Phys. A **29**, 11 (1996).
[8] Y. Kuramoto and H. Nakao, Phys. Rev. Lett. **76**, 4352 (1996); Y. Kuramoto and H. Nakao (to be published).
[9] A nonlinear stochastic process called the quadratic-noise Ornstein-Uhlenbeck process can be solved exactly by the Fokker-Plank equation, in which power law fluctuations realize in the steady state: see D. Gillespie, *Markov Processes* (Academic Press, Boston, 1992), p. 163.
[10] J. M. Deutsch, Physica (Amsterdam) **208A**, 433 (1994).
[11] W. Feller, *An Introduction to Probability Theory and Its Applications* (Wiley, New York, 1966).
[12] H. Takayasu, Phys. Rev. Lett. **63**, 2563 (1989); H. Takayasu, M. Takayasu, A. Provata, and G. Huber, J. Stat. Phys. **65**, 725 (1991).
[13] L. E. Reichl, *A Modern Course in Statistical Physics* (The University of Texas Press, Austin, Texas, 1980).
[14] A.-H. Sato and H. Takayasu (to be published).
[15] R. N. Mantegna and H. E. Stanley, Nature (London) **376**, 46 (1995); M. H. R. Stanley *et al.*, Fractals **3**, 415 (1996).