Control of High-Dimensional Chaos in Systems with Symmetry

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We demonstrate the successful control of a periodic orbit associated with two unstable manifolds in a system comprised of two coupled diode resonators. It is shown that due to symmetries generic to spatially extended systems a one-parameter control is not possible. A novel method of determining the local Liapunov exponents utilizing *orthogonal control* as well as geometric information is presented. [S0031-9007(97)03517-5]

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The ability to control unstable periodic orbits embedded within a chaotic system by applying small corrections to an accessible parameter was demonstrated in a seminal paper by Ott, Grebogi, and Yorke (OGY) [1]. Numerous examples of controlling low-dimensional chaos in physical systems [2] followed, but the experimental application of chaos control techniques to orbits exhibiting more than one unstable direction remains a formidable challenge. Extending the OGY method to high-dimensional systems with the ultimate goal to control spatiotemporal chaos proves a particularly difficult task. General theories for controlling unstable periodic orbits in n dimensions are presented in Ref. [3]. However, all of these are *one*-parameter schemes; multiparameter control is addressed in Ref. [4].

In this Letter we show that there exists a very important class of systems with multiple unstable directions for which one-parameter control algorithms generally will not succeed and that multiple controllers are necessary [5]. The assumed symmetries naturally arise in extended systems with spatial symmetries as well as in arrays of coupled oscillators. We therefore expect our results to be applicable to dynamical systems as diverse as reaction-diffusion equations [6], models of animal gait transitions [7], and synchronized chaotic oscillators [8].

We present experimental confirmation of these predictions for a system of two coupled diode resonators. The stabilization of a periodic orbit exhibiting two unstable, real eigenvalues was achieved implementing two independent controllers, since a one-parameter approach proved to be impossible. Following previous work [9], the volume in "feedback-gain-space" is mapped out, the boundaries of which correspond to curves of neutral stability. Along these boundaries, both (resulting) eigenvalues are of modulus one, and we are able to determine the local Liapunov exponents using geometrical information. We also present an alternative method in which control is applied along the (orthogonal) eigendirections of the system. Disabling one controller provides a direct way of measuring the Liapunov exponent along the respective eigenvector.

Experimental results.—The experimental results are obtained for two coupled diode resonators. A circuit diagram of this system, which has been previously studied in detail [10], is given in Fig. 1. It is comprised of a parallel combination of two diode resonators in series with a resistor, which allows for coupling and is driven sinusoidally. As the drive voltage is increased, the system period doubles once and then undergoes a Hopf bifurcation into a quasiperiodic state. The period doubling can lead to either an in-phase or a (symmetry breaking) out-of-phase period-2 orbit. Because of the particular way of coupling, our system strongly favors the latter one. A number of unstable low- and high-period orbits existing on top of the out-of-phase period-2 state have been successfully controlled [9,11]. We stress the qualitative difference between the control of these states and the homogeneous, in-phase orbits. The Hopf bifurcation and the associated complex eigenvalues do not occur for homogeneous, in-phase orbits.

We are targeting the period-1 state [12] which exhibits two real, unstable eigenvalues. The only accessible system parameters are the drive amplitude and the biases of the diodes. An OGY-based, one-parameter control scheme could therefore either globally perturb the *common* drive/bias [13] or apply perturbations to the bias of *one* of the individual resonators. Both choices are not feasible: Applying the control locally implies that the respective other resonator is controlled only through the coupling which in general is weak. When the

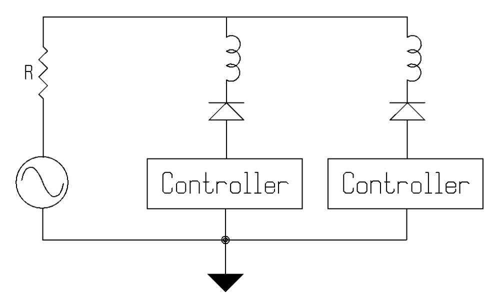

FIG. 1. The double diode resonator circuit. The coupling is provided by the resistor R. Each controller measures the deviation of the peak current through the diode from a set point and proportionally alters the respective dc bias every drive cycle.

common drive is chosen as control parameter, the fixed point moves along one of the eigenvectors, namely, the diagonal of the Poincaré section. It is well known that control schemes varying only one parameter can succeed only if its change affects all directions [9]. This "controllability condition" [14] will be violated by any coupled oscillator system which exhibits a similar symmetry as the diode resonators presented here. Control of the period-1 orbit was achieved by implementing two independent controllers. The control method used here is described in Ref. [15]. For each resonator, deviations from a respective set point are fed back to modulate the bias of the individual diode. Since there are two feedback strengths to vary, the pairs which lead to successful control fill a two-dimensional volume in "gain space." Intuitively, for uncoupled, identical resonators, the shape is expected to be a square. For finite coupling and not perfectly matched elements, the experimentally obtained data are given in Fig. 2(a). Note the lower curved corner along the diagonal, which is well described by the hyperbolic boundaries in the model (to be developed below). The agreement with the theory is seen to deteriorate at higher gains. The model predicts the less marked curvature in the upper corner as well as an early loss of control for high gains in the presence of noise and a slight mismatch in system parameters.

In our system it is straightforward to identify the two eigendirections [16]. The sum and the difference of the individual currents through the diodes correspond to the respective orthogonal projections of the two-dimensional state vector. Using this information, we implement *orthogonal control,* applying independent feedback control to each (one-dimensional) eigendirection. The experimental setup for this is shown in Fig. 3. The control signals are summed and subtracted and go through a variable gain stage to provide the in-phase and out-of-phase control signals. These are then combined to control the diode resonators.

In this way, the controllers are decoupled, which is reflected in the rectangular shape of the controlled area in Fig. 2(b). For perfectly matched resonators, one would expect a square instead of a rectangle. Simulations on the model confirm that even a slight mismatch in the individual elements results in a significant difference in the respective ranges over which control can be maintained.

Orthogonal control is also the basis for a novel, convenient method of determining the local Liapunov exponents. The data for Fig. 4 were collected via this approach by controlling both eigendirections, then disabling one controller and subsequently recording the system's trajectory. The rate of the exponential departure from the fixed point—contrived to be along an uncontrolled (unstable) eigenvector—yields the corresponding eigenvalue of the stability matrix. We find the slope associated with the out-of-phase period-2 state, $\lambda \sim 2.3$, to be significantly larger than the effective in-phase instability, $\lambda \sim 1.5$.

Coupled maps.—The diode resonator is known to be modeled quite well by a one-dimensional quadratic map [10]. The system of linearly coupled logistic maps [17]

$$x_{n+1} = r_1 x_n(1 - x_n) + \epsilon(y_n - x_n),$$
$$y_{n+1} = r_2 y_n(1 - y_n) + \epsilon(x_n - y_n)$$

models the resistively coupled pair of diode resonators remarkably well. On increasing the parameters r_i, it first period doubles and subsequently follows the quasiperiodic route to chaos. The coupling strength is proportional to ϵ, and the parameters r_i in the model correspond to the drive amplitude/individual bias of the diodes. For $r_1 = r_2 \equiv r$ the system is completely symmetric in its state variables, $\vec{x}_{n+1} = \vec{f}(\vec{x}_n, r_1, r_2) \equiv \vec{f}(\vec{x}_n, r)$. The Jacobian corresponding to the period-1 fixed point $\vec{x}_f = (x_f, x_f)$ reflects this symmetry and is of the form

$$\underline{\underline{J}} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad \text{with } a = r - \epsilon - 2rx_f, b = \epsilon .$$

The *real* eigenvalues and eigenvectors of this Jacobian [18] are $(a - b, a + b), ((-1, 1), (1, 1))$. The symmetry of the system will be prevalent if one would try to control the system using the common "drive," i.e., the common r as control parameter, in which case $\partial\vec{f}/\partial r$ will be parallel to one of the eigenvectors, $(\partial\vec{f}/\partial r) || (1, 1)$. In contrast, varying only one of the r_i's will shift the

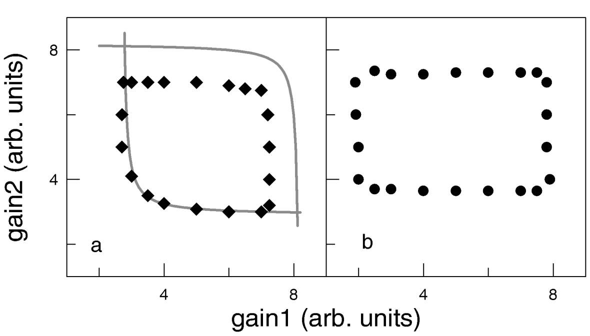

FIG. 2. Region over which control is maintained as a function of the two gains. (a) Each diode resonator was independently controlled using occasional proportional feedback control [15]. The coupling gives rise to hyperbolic boundaries. The shown hyperbolas (solid lines) are analytical results from the model assuming the eigenvalues from Fig. 1. (b) The two controllers are decoupled by *orthogonal control* resulting in a rectangular volume.

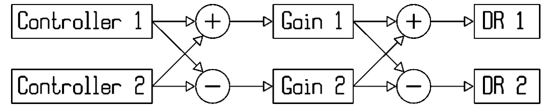

FIG. 3. Orthogonal control applied to the diode resonators (DR). The summing and subtracting of the control signals is depicted. The two resulting signals directly manipulate the two eigendirections.

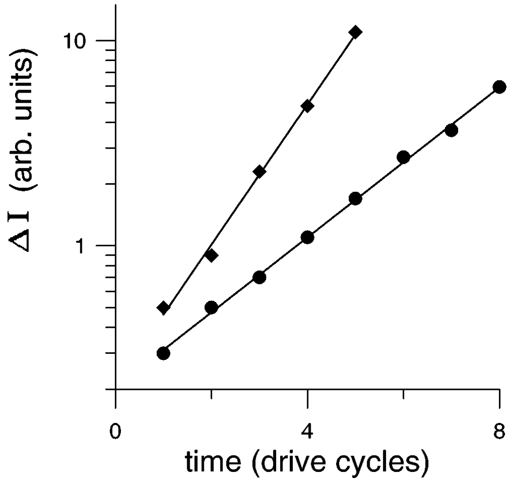

FIG. 4. Exponential departure of the peak current ΔI from the fixed point along the two unstable eigendirections. In each graph the system was constrained to one eigendirection through orthogonal control (see text), yielding eigenvalues/slopes of 2.3 (diamonds) and 1.5 (circles).

fixed point parallel to the coordinate axes, which is a "right" direction, i.e., not perpendicular to one of the eigenvectors, but the trade-off being that the second map will be controlled solely through the coupling which in general will be rather weak [19]. Therefore, the gain on the input from the 2nd controller will be very high, rendering the system highly susceptible to noise and measurement errors.

Employing two independent controllers in the experimental setup corresponds to varying the two r_i's independently, i.e., $\partial r_i^n = g_i \cdot [(\vec{x}_n)_i - x_f]$ with respective gains g_i. The modified Jacobian is

$$\tilde{\underline{\underline{J}}} = \begin{pmatrix} a - e_1 & b \\ b & a - e_2 \end{pmatrix}, \quad \text{with } e_i = x_f(1 - x_f) \cdot g_i$$

and with $\lambda_{1,2} = \frac{1}{2}[2a - e_1 - e_2 \pm \sqrt{4b^2 + (e_1 - e_2)^2}]$ as eigenvalues. The lines of constant λ's are hyperbolas [20]:

$$\lambda_{1,2} = c \Leftrightarrow e_2 = \hat{a} + \frac{b^2}{e_1 - \hat{a}}, \quad \text{with } \hat{a} = a - c\,.$$

The controllable area is therefore bounded by the two hyperbolas corresponding to $c = -1$ (upper boundary) and $c = 1$ (lower boundary). It can be shown that the volume is nonzero as long as $b < 1$.

A method of calculating the local instabilities in the case of a one-parameter control scheme was presented in Ref. [9]. We were also able to calculate the eigenvalues associated with the unstable period-1 orbit from the geometry of the controlled area in Fig. 2(a). Let (e_1, e_2) be one of the corners, and e_{small} and e_{big} the two intersections with the diagonal. Then we obtain for the two eigenvalues

$$\lambda_1 = 1 + 4\frac{B - B^2}{f(A,B)}, \qquad \lambda_2 = -1 - 4\frac{B - 1}{f(A,B)}, \quad (1)$$

where $A \equiv (e_1 - e_2)^2/(e_1 + e_2)^2, B \equiv e_{\text{big}}/e_{\text{small}}$ and $f(A,B) \equiv 1 + A - 2B + 2AB + B^2 + AB^2$. Note that A and B, and therefore the expressions for the λ_i's, depend only on the *ratios* of the gains e_i, and are thus independent of any specific scaling factors. Applying Eq. (1) to the experimental data from Fig. 2(a) yields $\lambda_{1,2} \sim (-1.9, -2.4)$ compared to $\lambda_{1,2} = (-1.5, -2.3)$ from Fig. 4. The rather poor agreement between the two different methods can be attributed to the aforementioned amplification of noise and system mismatches when high gains are employed.

Generalization to N dimensions.—It is well known that symmetries which lead to degenerate eigenvalues preclude one-parameter control schemes. In this Letter we showed that the presence of a different class of symmetries results in the need for multiple controllers. We will now prove that the controllability condition [14] will be violated for any system exhibiting such spatial symmetries. For the sake of simplicity we restrict ourselves to N coupled (identical) one-dimensional systems. Assume the local dynamics to be governed by some mapping function $x_{n+1} = f(x_n, r)$, with an adjustable parameter r and fixed point x_f. (Higher dimensional local elements can be treated in an analogous manner.) We exclusively consider *normalized* coupling functions $G(x^{i-\lambda}, \ldots, x^{i+\lambda}) \equiv G(i)$ with correlation length λ:

$$x_{n+1}^i = f(x_n^i, r^i) + G(x^{i-\lambda}, \ldots, x^{i+\lambda}) \equiv F_i(r^i, \vec{x}), \quad (2)$$

$i = 1, \ldots, N$, with the normalization property

$$\sum_k [\partial G(i)/\partial x_k]_{x_f} = \sum_k [\partial G(j)/\partial x_k]_{x_f} \equiv \text{const} \quad (3)$$

for all $i, j = 1, \ldots, N$. All systems exhibiting translational invariance obey this condition [21].

Any attempt to stabilize the uniform solution $x_n^i \equiv x_f$ involves the linearization of $\vec{F}$ around the fixed point. The special symmetry of coupling N identical elements is reflected in the resulting Jacobian $J_{ij} = [\partial f(x,r)/\partial x]\delta_{ij} + \partial G(x^{i-\lambda}, \ldots, x^{i+\lambda})/\partial x^j$, evaluated at the fixed point. Because of property (3), the sum of the row elements is equal for all rows:

$$\sum_k J_{ik} = \sum_k J_{jk} \equiv \text{const} \quad (4)$$

for all $i, j = 1, \ldots, N$. Thus, the N-dimensional diagonal vector $(1, 1, \ldots, 1)$ is one of the eigenvectors. If one were to choose the "common drive" as control parameter, i.e., globally perturb r, the fixed point would be shifted along the space diagonal, i.e., along one of the eigenvectors of the Jacobian

$$\frac{\partial \vec{F}}{\partial r} = \left(\frac{\partial f}{\partial r}, \ldots, \frac{\partial f}{\partial r}\right) \sim (1, 1, \ldots, 1)\,. \quad (5)$$

Therefore, the controllability condition [14] is violated for all systems with the symmetry properties (4) and (5).

In real experimental situations, the local elements will never be identical and (symmetry breaking) noise will always be present. Nevertheless, we expect $(\partial\vec{F}/\partial r)$ to be "almost" parallel to one of the eigenvectors, so that a one-parameter scheme would still be impractical.

In conclusion, we demonstrate that a one-parameter control in a very important class of high-dimensional dynamical systems is not possible. We experimentally stabilize a periodic orbit associated with two unstable manifolds utilizing two independent controllers. The results are confirmed using a model of two coupled maps and generalized to N dimensions.

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[12] The period-1 state is a special case of a general *homogeneous* period-n orbit.

[13] The effective drive amplitude is roughly proportional to $V_{\text{drive}}/V_{\text{bias}}$ [10].

[14] This condition can be formulated in terms of orthogonality: one-parameter control schemes can succeed only if $\vec{e}_i \cdot (\partial/\partial p)\vec{f}(x_f) \neq 0$ for all (unstable) contravariant eigenvectors $\vec{e}_i$ [9].

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[16] Again, this is a rather general consequence of the symmetries. In a variety of different systems it will be possible to easily identify the eigenvectors, as long as the Jacobian is a circulant matrix [22]. This renders orthogonal control broadly applicable.

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[18] The Jacobian of any *homogeneous* period-n fixed point will be of this form. This implies that only *real* eigenvalues are associated with homogeneous states, i.e., the Hopf bifurcation is possible only via an underlying *out-of-phase* period-2 fixed point. This particular Jacobian is a two-dimensional circular matrix [22] and thus fulfills this property (4).

[19] We remark that Ding *et al.* [3] *asymmetrically* coupled two Duffing oscillators in order to demonstrate their control technique. The coupling of the "uncontrolled" oscillator to the one which parameter was varied was twice as strong as the other way around.

[20] A similar result is obtained in the case of slightly different oscillators: Take the two diagonal elements to be different a's, namely, a_1 and a_2, then, the lines of constant λ's are still hyperbolas: $\lambda_{1,2} = c \Leftrightarrow e_2 = \hat{a}_2 + \frac{b^2}{e_1 - \hat{a}_1}$ with $\hat{a}_i = a_i - c$.

[21] The coupling function for, e.g., a diffusively coupled map lattice $x_{n+1}[i] = f(x_n[i]) + \epsilon(x_n[i-1] - 2x_n[i] + x_n[i+1])$ would be $G(i) = \epsilon(x_n[i-1] - 2x_n[i] + x_n[i+1])$, thus obeying property (3).

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