Consistency of the Baryon-Multimeson Amplitudes for Large-N_c QCD Feynman Diagrams

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We study the pion-baryon scattering process $\pi + B \rightarrow (n - 1)\pi + B$ in a QCD theory with a large number (N_c) of colors. It is known that this scattering amplitude decreases with N_c like $N_c^{1-n/2}$, and that its individual tree diagrams grow like $N_c^{n/2}$. The only way these two can be consistent is for $n - 1$ powers of N_c to be canceled when the Feynman diagrams are summed. We prove this to be true in tree order for any n. [S0031-9007(97)03541-2]

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QCD with a large number (N_c) of colors [1–3] is a useful tool and a beautiful theory. It allows strong interactions to be dealt with, thus complementing the understanding gained from lattice QCD calculations. In the infinite-color limit, special symmetry and simplifying dynamics often emerge, thus allowing a perturbative expansion in $1/N_c$ with impressive phenomenological results. Examples include predictions on baryon magnetic moments and mass splittings [4,5], large-N_c-improved heavy baryon and chiral Lagrangian expansions [6], and nucleon-nucleon potentials [7]. The only cloudy issue had been related to the consistency of the coupling between mesons and baryons, an issue which has been considerably clarified in recent years, thus lending credence to large-N_c phenomenological applications [4,5,8].

At first sight the physical attributes of these large-N_c baryons [2,3] look very different from the real ones, as well as the large-N_c mesons. They contain N_c quarks, whose various spin and isospin alignments produce a large number of baryon resonances, all with masses proportional to N_c. Since the emission of a pion may flip the spin and isospin of a quark, these resonances are coupled together into a multichannel problem. Moreover, it is known that the n-meson amplitude is proportional to $N_c^{1-n/2}$, in both the zero-baryon and the one-baryon sectors [2,3]. Thus all couplings between mesons are weak, decreasing as some powers of $1/\sqrt{N_c}$, but the Yukawa coupling of a pion to a baryon is strong and proportional to $\sqrt{N_c}$. This again marks the difference between large-N_c mesons and baryons.

The strength of this Yukawa coupling produces a number of serious problems. It implies that an n-meson tree diagram in the one-baryon sector is proportional to $N_c^{n/2}$, because this diagram contains n Yukawa coupling constants and because all baryon propagators are of $O(1)$. Not only does it generate undesirably large loop corrections, it utterly disagrees with the rule that an n-meson amplitude should decrease with N_c like $N_c^{1-n/2}$, for any n and for any number of loops [2,3]. Unless $n - 1$ powers of N_c are canceled in the sum of the $n!$ tree diagrams, the large-N_c rule in the one-baryon sector will not be self-consistent even in the tree approximation.

By demanding these cancellations to take place for $n = 2$ and $n = 3$, one obtains a set of conditions whose solution leads to interesting relations satisfied by physical baryons [4]. These are constraints consistent with quark-model results, but now obtained without the explicit assumptions of the model. In particular, it demands the presence of a tower of baryon resonances with equal spin J and isospin I, ranging in values from $\frac{1}{2}$ to $\frac{1}{2}N_c$ (assuming N_c to be odd). It has a rotational mass spectrum with a moment of intertia proportional to N_c. The presence of the baryon resonances is instrumental in effecting the cancellations needed for the consistency. Similar results were also obtained from the strong-coupling theory [9] and the Skyrme model [10].

Alternatively, one can derive the same physical relations from an explicit quark picture at large N_c [4], but then one must demonstrate these cancellations to take place for the sake of consistency. This has been carried out in the literature for $n = 2$ and $n = 3$ by direct calculations [4].

These cancellations are progressively more difficult to achieve for larger n because $n - 1$ powers of N_c must be canceled. To complicate the matter further, vertices for pion emissions are matrices coupling together all the baryon resonances. For these reasons it is not very hopeful to be able to demonstrate the cancellation for large n by straight forward computation in the usual way. However, by using a resummation technique recently developed [11,12], such cancellations can be established for *tree diagrams* very easily, and it is the purpose of this Letter to discuss how this is done.

The cancellation mechanism leading to the consistency is actually a rather general phenomenon, not confined to large-N_c QCD. It stems from a destructive interference of the multimeson amplitude, valid even when the mesons are off shell. It is this destructive interference that suppresses high powers of N_c, and it is the same destructive interference in high energy elastic scattering of quarks that suppresses high powers of $\ln s$ to enable the eikonal and the Regge pictures to be applied, and unitarity to be restored [12,13].

It should be noted, however, that the argument presented later is not sufficient to account for all the necessary cancellations in loop diagrams, so the consistency for loop amplitudes remains an open and challenging question. The cancellation for tree amplitudes may be viewed as a destructive interference between the external pions, brought on by their Bose-Einstein statistics. In loop amplitudes internal pions also participate in the interference but even so a sufficient amount of cancellation will not be attained. From the known examples of one-loop [3,4] and two-loop [14] cancellations in the one-pion sector, one sees that counterterms coming from renormalization are also necessary to effect the desirable cancellations. This somewhat complicates the physics and alters the combinatoric nature of the problem, which is why we are yet unable to extend our result to loop diagrams.

Non-Abelian cut diagrams.—The resummation mentioned above replaces the sum of Feynman tree diagrams [Fig. 1(a)] with the sum of *non-Abelian cut diagrams* [Fig. 1(b)] [11,12]. The latter are organized in such a way that the interferences of Bose-Einstein amplitudes are automatically built in. With that tool the proof of the consistency criterion follows almost immediately.

The resummation theorem applies to any tree amplitude whose main trunk carries a large energy, either in the form of a large mass as in the present case of large N_c, or a large kinetic energy as in the case of high-energy quark-quark elastic scattering. The energies and momenta of the emitted bosons are comparatively small, but they can be off shell to allow this tree amplitude to be sewed up to others to form a loop diagram. In this way the resummation theorem and the resulting non-Abelian cut diagrams are applicable in the presence of loops as well.

Tree diagrams will be labeled by the order their meson lines appear along the baryon trunk. The tree diagram in Fig. 1(a), for example, will be denoted by $[231465]$. We will construct from each Feynman tree diagram a *non-Abelian cut diagram* [12] by placing cuts on some of its propagators as follows. A cut is put after a meson line iff there is no meson to its right designated by a smaller number. Denoting a cut diagram by a subscript c, and indicating a cut by a vertical bar, the cut diagram for Fig. 1(a) is $[231465]_c = [231|4|65]$, as shown in Fig. 1(b).

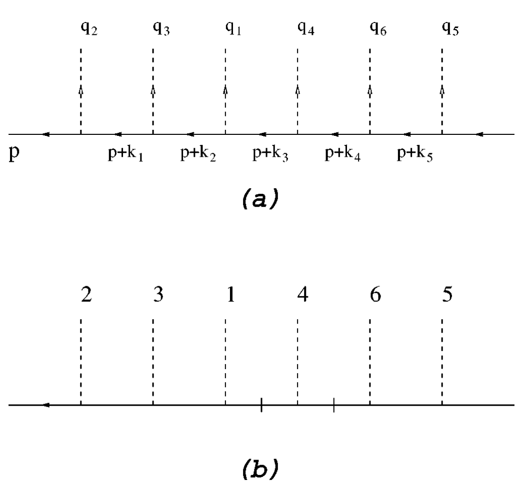

FIG. 1. (a) A Feynman tree diagram for baryon-meson scattering; (b) the non-Abelian cut diagram corresponding to (a).

Let p be the final momentum of the baryon and q_i be the outgoing meson momenta. Using $k_i = \sum_{j=1}^{i} q_{\sigma_j}$ to denote the sums of meson momenta for the diagram $[\sigma_1\sigma_2\cdots\sigma_n]$, the momentum of the ith baryon is then $p + k_i$. The assumption of the tree trunk carrying large energy is used to approximate the ith baryon propagator

$$\frac{M_i + \gamma(p + k_i)}{(p + k_i)^2 - M_i^2 + i\epsilon} \simeq \frac{1}{2}(1+\gamma^0)\frac{1}{k_i^0 - \Delta M_i + i\epsilon}, \tag{1}$$

where $\Delta M_i = M_i - M$ is the mass difference between the baryon resonance and the nucleon. Implicit in this is the assumption that while M and M_i are $O(N_c)$, the difference ΔM_i is at most $O(1)$ as $N_c \to \infty$ in order to keep all the baryons at a constant velocity.

With this approximation, the Feynman amplitude for $[\sigma_1\sigma_2\cdots\sigma_n]$ is given by

$$A[\sigma_1\sigma_2\cdots\sigma_n] = \frac{1}{2}(1+\gamma^0)a[\sigma_1\sigma_2\cdots\sigma_n] \times V[\sigma_1\sigma_2\cdots\sigma_n], \tag{2}$$

where $V[\sigma_1\sigma_2\cdots\sigma_n] = V_{\sigma_1}V_{\sigma_2}\cdots V_{\sigma_n}$ is simply the product of all the vertices V_i, and we have assumed that the projection operator $\frac{1}{2}(1+\gamma^0)$ commutes with the vertex operators V_i attached to the ith meson line. For the moment we will also assume that all $\Delta M_i = 0$, but this restriction will be lifted later. The spacetime part of the amplitude in (2) is then given by $a[\sigma_1\sigma_2\cdots\sigma_n] = -2\pi i\delta(\sum_{i=1}^{n} q_i^0)\prod_{i=1}^{n-1}(k_i^0 + i\epsilon)^{-1}$.

On the other hand, the amplitude for a *non-Abelian cut diagram* is defined as follows:

$$A[\sigma_1\sigma_2\cdots\sigma_n]_c = \frac{1}{2}(1+\gamma^0)a[\sigma_1\sigma_2\cdots\sigma_n]_c \times V[\sigma_1\sigma_2\cdots\sigma_n]_c, \tag{3}$$

where $a[\sigma_1\sigma_2\cdots\sigma_n]_c$ is obtained from $a[\sigma_1\sigma_2\cdots\sigma_n]$ by replacing the Feynman propagator $(k_i^0 + i\epsilon)^{-1}$ of a cut line by the Cutkosky cut propagator $-2\pi i\delta(k_i^0)$. The vertex part $V[\sigma_1\sigma_2\cdots\sigma_n]_c$ is obtained from $V[\sigma_1\sigma_2\cdots\sigma_n]$ by replacing the product of V_i's straddling *uncut lines* by (multiple) commutators. For example, $V[231465]_c = V[231|4|65] = [V_2,[V_3,V_1]]V_4[V_6,V_5]$.

The resummation formula (called the *multiple commutator formula* in [11]) asserts that the sum of the Feynman amplitudes is equal to the sum of the non-Abelian cut amplitudes,

$$\sum_{\sigma}^{n!} A[\sigma_1\sigma_2\cdots\sigma_n] = \sum_{\sigma}^{n!} A[\sigma_1\sigma_2\cdots\sigma_n]_c, \tag{4}$$

when the sum is taken over all the $n!$ permutations $\sigma = [\sigma_1\sigma_2\cdots\sigma_n]$ of $[12\cdots n]$.

In the special case when the vertices V_i are Abelian so they mutually commute, the only surviving term is the one without any commutator appearing, which is given by the cut diagram with every baryon propagator cut. The spacetime part $a[123\cdots n]_c$ is now a product of δ functions in q_i^0, showing neatly a very peaked interference pattern in all the variables q_i^0. Away from $q_i^0 = 0$, the interference is purely destructive.

In case of non-Abelian vertices, the different terms on the right-hand side of (4) carry different internal quantum numbers, and their spacetime parts exhibit varying degrees of destructive interference according to the number of δ functions present. However, since the number of δ functions plus the number of commutators is the same for every term, what is lacking in spacetime destructive interference is made up by the "destructive interference" in the internal quantum numbers, in the following sense. Imagine V_i to be the generators of a Lie group in a low-dimensional representation. Then products of V_i will contain progressively higher-dimensional representations and hence larger quantum numbers, but commutators of them will simply behave like a single V_i, creating only small quantum numbers. In this sense commutators represent an "interference" in which large quantum numbers tend to be wiped out. We shall see that it is this kind of interference that suppresses the high powers of N_c.

These formulas, suitably modified, are applicable even when baryon mass degeneracy is lifted, provided we insert into the tree new vertices $V_i' = \Delta M$ carrying away no energy. To see that, let ΔM be the diagonal operator whose matrix elements are the mass differences ΔM_i. Using the expansion

$$\frac{1}{k_i^0 - \Delta M + i\epsilon} = \frac{1}{k_i^0 + i\epsilon} \sum_{m=0}^{\infty} \left(\frac{\Delta M}{k_i^0 + i\epsilon} \right)^m, \qquad (5)$$

we see that the new vertex necessary is simply $V_i' = \Delta M$.

Proof of the consistency criterion.—Let us first review some standard facts in the one-baryon sector [5]. The quark-pion interaction is proportional to $N_c^{-\frac{1}{2}} \bar{\psi} \gamma^\mu \gamma_5 \tau^a \partial_\mu \pi_a$, in which the coefficient $N_c^{-\frac{1}{2}}$ is fixed (see later) by the requirement of the meson-baryon Yukawa constant being of order $\sqrt{N_c}$. In the rest frame of the baryon, the large component ϕ of the Dirac spinor ψ dominates, so this interaction is reduced to an expression proportional to $N_c^{-\frac{1}{2}} \{\sigma^i \tau^a\} \partial_i \pi_a$, where $\{\Gamma\} \equiv \phi^\dagger \Gamma \phi$. This in turn determines the pion-baryon vertex to be proportional to $V_i = N_c^{-\frac{1}{2}} \{\sigma^i \tau^a\}$, with "$a$" labeling the isospin of the pion it couples to.

The large-N_c rules for n-meson amplitude in the one-baryon sector were derived from the quark picture using the Hartree approximation [2]. In this approximation, the wave function of a baryon state $|B_{J,I}\rangle$ with spin J and isospin I can be represented by an $SU(2)_J$ and an $SU(2)_I$ Young tableau, as shown in Figs. 2(a) and 2(b). For this color-singlet and s-state baryon to have

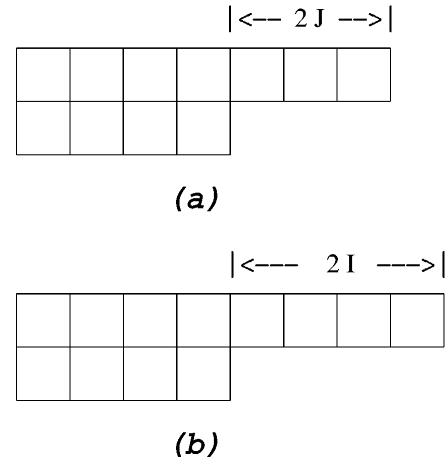

FIG. 2. Young tableaux for (a) $SU(2)_J$ and (b) $SU(2)_I$.

a totally antisymmetric wave function, the spin and isospin tableaux must be identical, which implies $I = J$. Incidentally, if there are more than two flavors, then only the light-quark spins match this way. Adding to that the spins of the strange and other quarks, the allowed baryon spin now takes on many values, thus forming a considerably larger multiplet for a given I.

Returning to the case of two flavors, the double boxes appearing in a column of the tableau are singlets of quark pairs in J or I, so they are killed by the spin operator $\{\sigma^i\}$ and the isospin operator $\{\tau^a\}$. However, $\{\sigma^i \tau^a\} \neq \{\sigma^i\}\{\tau^a\}$, so these singlets are not killed by $\{\sigma^i \tau^a\}$. Since there are $O(N_c)$ columns in a tableau, the baryon matrix element $\langle V_i \rangle$ is of the order $N_c^{-\frac{1}{2}} N_c = \sqrt{N_c}$, as it should for a Yukawa coupling constant. For simplicity, the notation $\langle \mathcal{O} \rangle \equiv \langle B_{J',I'} | \mathcal{O} | B_{J,I} \rangle$ has been used.

Similarly, the matrix element of a two-body operator, $\langle \{\sigma^i \tau^a\}\{\sigma^j \tau^b\} \rangle$, is of order N_c^2, but their commutator is only of order N_c. This is so because $[\{\Gamma_1\}, \{\Gamma_2\}] = \{[\Gamma_1, \Gamma_2]\}$, and because

$$[\sigma^\mu \tau^\alpha, \sigma^\nu \tau^\beta] = \frac{1}{2}[\sigma^\mu, \sigma^\nu][\tau^\alpha, \tau^\beta] + \frac{1}{2}[\sigma^\mu, \sigma^\nu] + [\tau^\alpha, \tau^\beta]_+ \qquad (6)$$

is a linear combination of $\sigma^\lambda \tau^\gamma$, thus making its matrix elements order N_c. This means that $\langle [V_i, V_j] \rangle$ is of order $(N_c^{-\frac{1}{2}})^2 N_c = 1$. Similarly, each time an additional commutator appears, the matrix element is reduced by an additional power of $N_c^{-\frac{1}{2}}$. In particular, the matrix element of an n-tuple commutator is of order $N_c^{1-n/2}$. In these expressions, σ^0 and τ^0 are, respectively, the unit matrices in the spin and isospin spaces.

We proceed now to prove the consistency criterion for the pion-baryon scattering amplitude $\pi + B \rightarrow (n-1)\pi + B$. We shall first assume all $\Delta M_i = 0$ and all pions to be coupled directly to the baryon. We will also take the pion mass to be nonzero.

One of the n pions is incoming and the remaining $n-1$ are outgoing, so one of the q_i^0 is negative but the rest of them are positive. Energy conservation, or the requirement of the initial baryon to be on shell, demands that the sum of the n q_i^0's be zero. However, because all but one of them are positive, a partial sum of them can never be zero, which is to say that the only surviving terms in (4) are the ones without any Cutkosky cut. These incidentally are the non-Abelian cut diagrams with pion 1 at the far right. For such terms, the vertex factor $V[\sigma_1\sigma_2\cdots\sigma_n]$ contains n-tuple commutators of the vertices V_i, whose baryon matrix elements are of order $N_c^{1-n/2}$ as we saw before. This then shows that the sum of the $n!$ tree diagrams is of order $N_c^{1-n/2}$, and we have attained just the right amount of cancellations required by consistency.

This conclusion remains valid without the special assumptions. If $\Delta M_i \neq 0$, then rotational invariance demands it to be of the form $\Delta M = c\{\vec{\sigma}\}\cdot\{\vec{\sigma}\}/N_c$ [5], so $[\Delta M, \{\sigma^i\tau^a\}] = (2c/N_c)\{\sigma^i\tau^a\}$. This means that any commutator with ΔM will only lead to subleading dependences at large N_c. The same will also be true if some of the mesons are coupled directly to other mesons rather than the baryon, because all meson couplings vanish as a power of $N_c^{-\frac{1}{2}}$. Seagull type of diagrams are also negligible.

Before ending, we also remark on the special situation when the pion is massless. In that case pion energies can be zero and the δ functions in the partial sums of q_i^0 can no longer be thrown away so easily. These terms have less commutators of V_i and hence higher powers of N_c than $N_c^{1-n/2}$. However, these correspond exactly to the terms in which different pions hit different quarks [2,3], rather than the same quark which leads to the familiar dependence of $N_c^{1-n/2}$.

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