Extraction of Spin-Orbit Interactions from Phase Shifts via Inversion

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An exact inversion procedure for obtaining the central and spin-orbit potential from phase shifts at fixed energy is described. The method, based on Sabatier interpolation formulas, reduces the nonlinear problem to linear-algebraic equations. We have tested the method with a Woods-Saxon potential with a strong spin-orbit component. [S0031-9007(97)03688-0]

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The problem of constructing the interaction between two particles, whether with or without spin, from scattering data at fixed energy has been extensively investigated in the past three decades due to its great significance in diverse fields ranging from nuclear [1], atomic [2], and molecular [3] physics. There are two different approaches to the problem. The most common one is the local inversion approach which is either a phenomenological potential-parameter fitting [4] or an iterative perturbation method [5]. With the local inversion approach, one starts from some initial guess and the results are often dependent on the starting point. The questions of existence, uniqueness, and ill-posedness are simply ignored. A more rigorous and global approach is to use inverse scattering theory (IST) [6], which establishes a connection between the potential and the scattering function by a certain linear equation [7]. An important tool in IST for fixed energy is a set of coupled, linear-algebraic equations called the Regge-Newton equations or the Newton interpolation formula [8]. By solving these equations, one can construct the central potential from the scattering function directly for spinless scattering systems. For the spin-orbit interaction, an analog of the Regge-Newton equations known as the Sabatier interpolation formulas was first found by Sabatier [9]. However, the Sabatier interpolation formulas are nonlinear and not readily useful for inversion analysis. Hooshyar [10] developed a scheme which brings the Sabatier interpolation formulas closer to realistic applications. Hooshyar's procedure is rather complicated as it requires solutions to coupled, linear integral equations and a nonlinear Reccati equation, apart from a set of infinite coupled linear-algebraic equations.

An approximate inversion method for the spin-orbit interaction has been proposed recently by Leeb *et al.* [11]. Their procedure, based on the second-order distorted wave Born approximation, allows the separation of the central and the spin-orbit potentials from the phase shifts. After the separation, one can employ any inversion method for the central potential on two decoupled sets of phase shifts. The procedure works well for weak spin-orbit interactions and high energies. This scheme has been applied to nucleon-alpha spin-orbit potentials at low energies [12].

In this Letter, we present an exact inversion scheme for the spin-orbit interaction. The main achievement of this scheme is the transformation of the Sabatier interpolation formulas, which are nonlinear, into finite, coupled linear-algebraic equations. The scheme is computationally simple as the whole procedure requires only solutions to linear-algebraic equations. In this scheme, we assume that the central and the spin-orbit potentials are known from a certain radial distance to infinity. This assumption is similar to the one made in the modified Newton method for the inversion of central potentials [14] and usually does not pose any difficulty in practice.

Before presenting our method, we note a very recent paper by Eberspächer *et al.* [13] reporting an extended and modified Newton-Sabatier method for the coupled-channel case. Unlike the spin-orbit interaction which depends linearly on the angular momentum, the potential for the coupled-channel scattering is a matrix independent of the angular momentum. It is hoped that in the future our inversion scheme for the spin-orbit interaction and Eberspächer's inversion scheme for the potential matrix can be integrated together to apply to coupled-channel scattering systems with spin.

Now let us consider the scattering of a spin–$\frac{1}{2}$ particle by a central and a spin-orbit field. The potential has the form

$$V(r) = E_{\mathrm{cm}}[U_c(r) + 2\mathbf{l}\cdot\mathbf{s}U_s(r)]. \tag{1}$$

Since the energy is fixed, it is convenient to introduce $\rho = kr$. The dimensionless, reduced radial Schrödinger equation is given by

$$\left[\frac{d^2}{d\rho^2} + 1 - U_c + \frac{1}{2}U_s \mp \lambda U_s - \frac{\lambda^2 - (1/4)}{\rho^2}\right]\chi_\lambda^{\pm}(\rho) = 0, \tag{2}$$

with $\lambda = \ell + 1/2$. Sabatier gives an interpolation formula [9] for the regular solutions $\chi_\lambda^{\pm}(\rho)$ for U_c and U_s:

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$$\chi_\lambda^\pm(\rho) = F^\pm(\rho)\psi_\lambda(\rho) + \sum_{\mu\in S} \frac{\psi_\mu(\rho)\psi'_\lambda(\rho) - \psi_\lambda(\rho)\psi'_\mu(\rho)}{\lambda^2 - \mu^2}\Big[b_\mu\chi_\mu^\pm(\rho) - a_\mu^\pm\chi_\mu^\mp(\rho)\Big]\frac{2\mu}{\pi}, \tag{3}$$

with $S = \{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots\}$ and

$$F^\pm(\rho) = \exp\Big[\pm\int_0^\rho \frac{t}{2}U_s(t)\,dt\Big] = \frac{2}{\pi\rho}\,a_0\chi_0(\rho)\psi_0(\rho) + \frac{2}{\pi\rho}\sum_{\mu\in S}\Big[a_\mu^\pm\chi_\mu^\mp(\rho) + b_\mu\chi_\mu^\pm(\rho)\Big]\psi_\mu(\rho). \tag{4}$$

The functions $\psi_\lambda(\rho)$ are the regular solutions [with $\psi_\lambda(\rho) \longrightarrow (\frac{1}{2}\pi\rho)^{1/2}(\frac{\rho}{2})^\lambda/\Gamma(1+\lambda)$ as $\rho \longrightarrow 0$] for a given reference central potential U_0 and b_λ are determined by U_0. For $U_0 = 0$, b_λ are simply

$$b_\lambda = \begin{cases} 1 & \text{for positive integer } \lambda \\ 0 & \text{otherwise} \end{cases} \lambda \in S. \tag{5}$$

Note that Eqs. (3) and (4) involve physical values of λ $\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}$ as well as nonphysical values of λ $\{0, 1, 2, \ldots\}$. From Eq. (4), the spin-orbit potential is given by

$$U_s = \pm\frac{2}{\rho}\,\frac{(F^\pm)'}{F^\pm}, \quad \text{with } F^+F^- = 1. \tag{6}$$

Defining

$$G^\pm(\rho) = \frac{2}{\pi\rho}\sum_{\mu\in S}\mu\Big[a_\mu^\pm\chi_\mu^\mp(\rho) - b_\mu\chi_\mu^\pm(\rho)\Big]\psi_\mu(\rho), \tag{7}$$

U_c can be expressed by

$$U_c = U_0 + \frac{1}{2}U_s - \frac{1}{\rho}(G^+F^- + G^-F^+)' + \frac{\rho^2}{4}U_s^2. \tag{8}$$

From Eqs. (6) and (8), it can be seen that both U_c and U_s vary as r^{-1} towards the origin. Let us assume that the reference potential is a Coulomb potential with Coulomb phase shifts σ_λ and Coulomb parameter η. The wave functions then have the asymptotic forms as $\rho \longrightarrow \infty$:

$$\psi_\lambda(\rho) = \sin\Big[\rho - \frac{1}{2}\Big(\lambda - \frac{1}{2}\Big)\pi + \sigma_\lambda - \eta\ln 2\rho\Big], \tag{9}$$

$$\chi_\lambda^\pm(\rho) = c_\lambda^\pm\sin\Big[\rho - \frac{1}{2}\Big(\lambda - \frac{1}{2}\Big)\pi + \delta_\lambda^\pm - \eta\ln 2\rho\Big]. \tag{10}$$

The task for the inversion is first to find the potential coefficients $a_\lambda^\pm$ from the phase shifts $\delta_\lambda^\pm$ for the full potential and then to determine the wave functions $\chi_\lambda^\pm$ from the potential coefficients $a_\lambda^\pm$. Once $a_\lambda^\pm$ and $\chi_\lambda^\pm$ are determined, U_s can be constructed via $F^\pm$. Similarly one can obtain U_c via $F^\pm$ and $G^\pm$. The Sabatier interpolation formula, Eq. (3), does not give the solution because the functions $F^\pm(\rho)$ are unknown and sought. In the following, we show how we find $a_\lambda^\pm$ and $\chi_\lambda^\pm$.

Assuming that $\triangle U(= U_c - U_0)$ and U_s are zero for $\rho > \rho_0$, then $\chi_\lambda^\pm(\rho)$ is a linear combination of the regular and irregular solutions, denoted as $\mathcal{F}_\lambda(\rho)$ and $\mathcal{G}_\lambda(\rho)$, respectively, of the reference potential U_0. So we have for $\rho > \rho_0$,

$$\psi_\lambda(\rho) = \mathcal{F}_\lambda(\rho), \tag{11}$$

$$\chi_\lambda^\pm(\rho) = c_\lambda^\pm[\cos(\delta_\lambda^\pm - \sigma_\lambda)\mathcal{F}_\lambda(\rho) + \sin(\delta_\lambda^\pm - \sigma_\lambda)\mathcal{G}_\lambda(\rho)], \tag{12}$$

$$F^\pm(\rho) = \exp\Big[\pm\int_0^{\rho_0}\frac{t}{2}U_s(t)\,dt\Big] = h^\pm \quad \text{with } h^+h^- = 1. \tag{13}$$

Our assumption also makes all phase shifts $\delta_\lambda^\pm$ with $\lambda > \lambda_{\max}$ negligible. $\lambda_{\max}$ is related to ρ_0 and the strength of the potential. So S becomes a finite set: $S = \{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots, \lambda_{\max}\}$. Using the following abbreviations:

$$A_\lambda^\pm = a_\lambda^\pm c_\lambda^\mp h^\mp, \tag{14}$$

$$C_\lambda^\pm = c_\lambda^\pm h^\mp, \tag{15}$$

$$T_\lambda^\pm(\rho) = \cos(\delta_\lambda^\pm - \sigma_\lambda)\mathcal{F}_\lambda(\rho) + \sin(\delta_\lambda^\pm - \sigma_\lambda)\mathcal{G}_\lambda(\rho), \tag{16}$$

$$W_{\lambda\mu}(\rho) = \frac{\psi_\mu(\rho)\psi'_\lambda(\rho) - \psi_\lambda(\rho)\psi'_\mu(\rho)}{\lambda^2 - \mu^2}, \tag{17}$$

Eq. (3) gives $4\lambda_{\max} + 2$ linear equations for $A_\lambda^\pm$ and $C_\lambda^\pm$ at $\rho > \rho_0$, with $\lambda \in \{0\} \cup S$:

$$\mathcal{F}_\lambda(\rho) = C_\lambda^\pm T_\lambda^\pm(\rho) + \sum_{\mu\in S}\frac{2\mu}{\pi}W_{\lambda\mu}(\rho)[T_\mu^\mp(\rho)A_\mu^\pm - b_\mu T_\mu^\pm(\rho)C_\mu^\pm]. \tag{18}$$

Also Eq. (4) gives

$$\frac{\pi\rho}{2} = A_0^{\pm} T_0(\rho)\mathcal{F}_0(\rho) + \sum_{\mu\in S}[T_\mu^{\mp}(\rho)A_\mu^{\pm} + b_\mu T_\mu^{\pm}(\rho)C_\mu^{\pm}]\mathcal{F}_\mu(\rho). \tag{19}$$

To determine the $8\lambda_{\max} + 2$ coefficients $A_\lambda^{\pm}$ and $C_\lambda^{\pm}$ (except for $A_0^{\pm}$), Eq. (18) may be solved at two radial distances ρ_0 and $\rho_1 (> \rho_0)$. Then $A_0^{\pm}$ can be determined using Eq. (19). The coefficients $A_\lambda^{\pm}$ and $C_\lambda^{\pm}$ obtained from Eqs. (18) and (19) depend on ρ_0 and ρ_1. To reduce the dependency of ρ, one may use a linear least-squares method for the overdetermined system, Eqs. (18) and (19) at N (>2) points. Also we have found that Eq. (18) is a highly unstable system and the coefficients $C_\lambda^{\pm}$ for large λ tend to have very high values. We suggest a way to solve the problem is the following. Letting $\rho \longrightarrow \infty$, Eq. (18) becomes

$$1 = C_\lambda^{\pm}\exp(i\delta_\lambda^{\pm}) + \sum_{\mu\in S}[A_\mu^{\pm}\exp(i\delta_\mu^{\mp}) - C_\mu^{\pm} b_\mu \exp(i\delta_\mu^{\pm})]\exp\left[i\frac{\pi}{2}(\lambda - \mu)\right]L_{\lambda\mu}, \tag{20}$$

with

$$L_{\lambda\mu} = \frac{2\mu}{\pi}\,\frac{\sin\frac{\pi}{2}(\lambda - \mu)}{\lambda^2 - \mu^2}(1 - \delta_\mu^\lambda) + \frac{1}{2}\,\delta_\mu^\lambda. \tag{21}$$

The real and imaginary parts of Eq. (20) are

$$\cos\delta_\lambda^{\pm} = C_\lambda^{\pm} + \sum_{\mu\in S} A_\mu^{\pm}\cos\left[(\delta_\mu^{\mp} - \delta_\lambda^{\pm}) + \frac{\pi}{2}(\lambda - \mu)\right]L_{\lambda\mu} - C_\mu^{\pm} b_\mu \cos\left[(\delta_\mu^{\pm} - \delta_\lambda^{\pm}) + \frac{\pi}{2}(\lambda - \mu)\right]L_{\lambda\mu}, \tag{22}$$

$$\sin\delta_\lambda^{\pm} = \sum_{\mu\in S} -A_\mu^{\pm}\sin\left[(\delta_\mu^{\mp} - \delta_\lambda^{\pm}) + \frac{\pi}{2}(\lambda - \mu)\right]L_{\lambda\mu} + C_\mu^{\pm} b_\mu \sin\left[(\delta_\mu^{\pm} - \delta_\lambda^{\pm}) + \frac{\pi}{2}(\lambda - \mu)\right]L_{\lambda\mu}. \tag{23}$$

We have found that the use of one of Eqs. (22) and (23) together with Eq. (18) gives much more stable solutions. The reason is that Eq. (20) provides an exact relation between $A_\lambda^{\pm}$ and $C_\lambda^{\pm}$. Once $A_\lambda^{\pm}$ and $C_\lambda^{\pm}$ are found, the $a_\lambda^{\pm}$ are determined from Eqs. (14) and (15) with the use of $(h^{\pm})^2 = C_0^{\mp}/C_0^{\pm}$.

With $a_\lambda^{\pm}$ determined, we next find $\chi_\lambda^{\pm}(\rho)$ for all ρ and for $\lambda \in \{0\} \cup S$. Inserting Eq. (4) into Eq. (3) gives $4\lambda_{\max}$ homogeneous linear equations for $\chi_0(\rho)$ and $4\lambda_{\max}$ values of $\chi_\lambda^{\pm}(\rho)$ with $\lambda \in S$

$$\mathcal{F}_0(\rho)\mathcal{F}_\lambda(\rho)a_0\chi_0(\rho) = \sum_{\mu\in S}\{[\mu\rho W_{\lambda\mu}(\rho) - \mathcal{F}_\lambda(\rho)\mathcal{F}_\mu(\rho)]a_\mu^{\pm}\chi_\mu^{\mp}(\rho) + [\delta_{\mu\lambda}\pi\rho 2^{-1} - b_\mu\mathcal{F}_\lambda(\rho)\mathcal{F}_\mu(\rho) - \mu\rho b_\mu W_{\lambda\mu}(\rho)]\chi_\mu^{\pm}(\rho)\}. \tag{24}$$

Equation (24) defines $\chi_\lambda^{\pm}$ up to an arbitrary constant. To convert Eq. (24) into an inhomogeneous equation, we divide both sides of the equation by χ_0. Then we obtain $\chi_\lambda^{\pm}/\chi_0$ for each ρ. χ_0 can be determined by putting $\chi_\lambda^{\pm}/\chi_0$ into Eq. (4) and using the relation $F^+F^- = 1$. With the absolute values of $\chi_\lambda^{\pm}$ and $a_\lambda^{\pm}$ for $\lambda \in \{0\} \cup S$, we construct first U_s from Eq. (6) and then U_c from Eq. (8).

The method just described is applicable for complex potentials. For simplicity, we tested the method with a real Woods-Saxon potential:

$$V_c(r) = -V_0\left[1 + \exp\left(\frac{r - R}{a}\right)\right]^{-1}, \tag{25}$$

$$V_{so}(r) = 2V_{so}\frac{1}{r}\,\frac{d}{dr}\left[1 + \exp\left(\frac{r - R}{a}\right)\right]^{-1}, \tag{26}$$

with the parameters

$$V_0 = 50\ \text{MeV}, \quad V_{so} = 15\ \text{MeV}, \quad R = 3\ \text{fm}, \quad a = 0.6\ \text{fm}, \tag{27}$$

and with the reference potential being zero. The phase shifts ($\lambda_{\max} = 40.5$) from the Woods-Saxon potential at $E_{\text{cm}} = 150$ MeV have been used as input to our procedure. ρ_0 and ρ_1 have been set to 26 and 26.3, respectively. The inverted central and spin-orbit potentials are shown in Fig. 1. We can see from the figure that the reproduction is quite good from 2 fm outward. For radius smaller than 2 fm, the inverted potentials are oscillatory. The oscillatory behavior exists in inversion methods employing interpolation formulas like the Newton method [15], the modified Newton method [14], and the coupled-channel modified Newton method [13]. In fact, the oscillations are relatively small in our test case compared to those from the other methods. At the moment, we are investigating ways to further reduce these oscillations.

In conclusion, we have demonstrated a procedure for an exact spin-orbit inversion. Applications of the method to experimental data are under investigation. The experimental measureables are the unpolarized differential cross section and polarization data (Sherman function or analyzing power), not the phase shifts. Obtaining the phase shifts from the the experimental measureables is an important issue from the inversion point of view. For elastic scattering, the generalized unitarity theorem provides a way to extract phase shifts from the differential cross section and polarization data [16]. In the near future we will report our inversion analysis on electron-Xenon

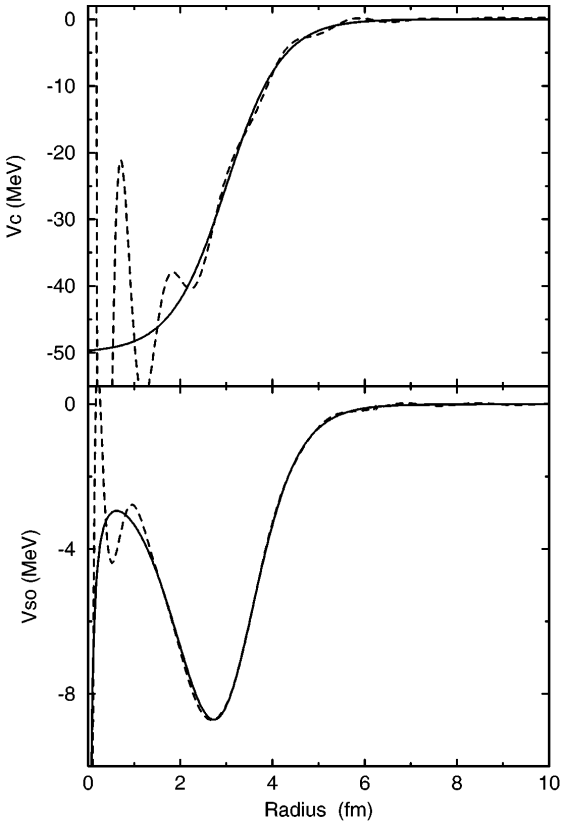

FIG. 1. The comparison of the inverted (dashed curves) and original (solid curves) potentials with central V_c and spin-orbit V_{so} components at $E_{cm} = 150$ MeV.

elastic scattering at 5 eV. It will be the first inversion analysis in which the interaction with central and spin-orbit components is deduced from experimental data without using any free parameters.

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