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Balian-Low Theorem for Landau Levels

J. Zak

Department of Physics, Technion-Israel Institute of Technology, Haifa, Israel 32000

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The Balian-Low theorem is applied to the motion of an electron in the *xy* plane with a magnetic field *B* perpendicular to this plane. The energy spectrum for this problem is the Landau levels. It is shown that the eigenfunctions for the Landau levels cannot be chosen sufficiently localized in order to make both uncertainties Δx and Δy finite. A similar result holds for the coordinates of the orbit center. With this restriction on the localization, complete orthonormal sets are defined on von Neumann lattices. [S0031-9007(97)03641-7]

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The Balian-Low [1,2] theorem relates to a set of orthogonal states on a von Neumann lattice. This is a rectangular lattice in the phase plane with a unit cell of area *h*. Such a lattice was first introduced by von Neumann [3] in the *xy* plane and independently by Gabor [4] in the time-frequency plane. The Balian-Low theorem can be formulated in the following way. Given a square integrable function $\psi(x)$, one builds out of it by translations in the phase plane a von Neumann set $\psi_{mn}(x)$, where the indices m, n label the number of the unit cell on the von Neumann lattice in the phase plane $[\psi_{00}(x) \equiv \psi(x)]$. m and n assume all integer values $0, \pm 1, \ldots$. The theorem then claims that if $\psi_{mn}(x)$ are orthogonal to $\psi_{m'n'}(x)$ for all $m, n \neq m', n'$ it follows that at least one of the two quantities $\langle x^2 \rangle$ or $\langle p^2 \rangle$ in the state $\psi(x)$ diverges (the triangular brackets denote the expectation value). Although discovered by physicists, the Balian-Low theorem has become of widespread interest in the engineering literature of signal processing [5].

In physics the von Neumann lattice has been known to appear in a natural way in the motion of an electron in a constant magnetic field $\vec{B}$ [6]. In particular, this has to do with the construction of magnetic Wannier functions [7,8]. There is an interesting connection between the Balian-Low theorem and the properties of magnetic Wannier functions in the *xy* plane for the case when $\vec{B}$ is parallel to the *z* axis, despite the fact that the latter problem has 2 degrees of freedom. This can be seen in the following way. Instead of the pairs of the conjugate coordinates xp_x and yp_y one can choose two other pairs: the x and y components of the velocity v_x, v_y (which will later be connected to the operators P_W and W, respectively) and the components X, Y of the magnetic orbit center. The Hamiltonian does not depend on the *XY* degree of freedom. They are constants of motion, and they determine the infinitesimal magnetic translations [9]. Correspondingly, the commuting finite translations for X and Y create a von Neumann lattice in the phase plane of the orbit center [6]. These same translations when considered in the *xy* plane are nothing else but the commuting magnetic translations (translations accompanied by gauge shifts) [9], which are used in the construction of magnetic Wannier functions [10–12]. As was already mentioned when only a magnetic field is present and there is no periodic potential, the Hamiltonian does not depend on the coordinates X and Y of the orbit center [13]. Each Landau level is then infinitely degenerate with respect to the location of this center and one can therefore apply the Balian-Low theorem to it.

In this Letter we show how the Balian-Low theorem can be directly applied to the wave functions of a single Landau level. When the motion is restricted to the *xy* plane perpendicular to the magnetic field $\vec{B}$ the energy spectrum consists of discrete levels which carry the name of Landau [14]. The wave function for each Landau level ℓ can be written either as $\psi_\ell(x, y)$ or $\phi_\ell(W, X)$.

In the latter form it separates into a product function with one factor $\gamma_\ell(W)$ depending on W [its relation to v_y is given in Eq. (11)] and the other one $\chi(X)$ on X only. Since the Hamiltonian does not depend on the orbit center X, the factor $\chi(X)$ in the wave function can be chosen completely arbitrarily and one can then apply to the XY degree of freedom the Balian-Low theorem. This is done by applying to $\chi(X)$ the complete commuting set of translations in the XY plane. They create the von Neumann set, and an explicit condition is given on $\chi(X)$ for the set to be orthogonal. For this purpose the kq representation [12] turns out to be convenient. Having carried everything out in the WX coordinates, one can then transform the results into the xy plane. An analysis of the Balian-Low theorem for an arbitrary Landau level is presented.

We start by formulating the Balian-Low theorem in the XY plane of the orbit center. For this we replace Y by $P_X = (\hbar/\lambda^2)Y$ with $\lambda^2 = \hbar c/eB$ being the square of the cyclotron radius. X and P_X are a couple of conjugate coordinates with $[X, P_X] = i\hbar$. The shift operator $D(\alpha)$ is used [15]

$$D(\alpha) = \exp(\alpha a^+ - \alpha^* a), \tag{1}$$

where a is the annihilation operator

$$a = \frac{1}{\lambda\sqrt{2}}\left(X + \frac{i}{\hbar}\lambda^2 P_X\right) \tag{2}$$

and α is a number

$$\alpha = \frac{1}{\lambda\sqrt{2}}\left(\overline{X} + \frac{i}{h}\lambda^2\overline{P}_X\right). \tag{3}$$

Here $\overline{X}$ and $\overline{P}_X$ are expectation values of the operators X and P_X in the eigenstate of a. a^+ is the Hermitian conjugate of a and α^* is the complex conjugate of α. A von Neumann set in the XP_X phase plane is [3]

$$\chi_{mn}(X) = D(\alpha_{mn})\chi(X), \tag{4}$$

where α_{mn} is obtained from α in Eq. (3) by setting $\overline{X} = md$, $\overline{P}_X = \hbar(2\pi/d)$ with d a constant. $\chi(X)$ in Eq. (4) is an arbitrary square integrable function. The properties of the von Neumann set in Eq. (4) are determined by the initial function $\chi(X)$. The Balian-Low theorem states that if the functions in Eq. (4) are orthogonal for $mn \neq m'm'$, then at least one of the uncertainties ΔX or ΔP_X in the $\chi(X)$ state diverges. An example of a normalized χ function that leads to an orthogonal von Neumann set in Eq. (4) is

$$\chi(X) = \begin{cases} \frac{1}{\sqrt{d}}, & |X| < d/2 \\ 0, & |X| > d/2 . \end{cases} \tag{5}$$

For the χ in Eq. (5), $\Delta X = d/\sqrt{12}$ and $\Delta P_X = \infty$, in agreement with the Balian-Low theorem.

The condition on χ for an orthogonal von Neumann set [Eq. (4)] was first formulated in the kq representation [6] (see also Ref. [1]). This condition is that the absolute value of the kq function, $C(k,q)$, is constant [16]

$$|C(k,q)| = \text{const}, \tag{6}$$

where (a is an arbitrary constant) $C^{(a)}(k,q)$ and $\chi(X)$ are related in the following way [12]

$$C^{(a)}(k,q) = \left(\frac{a}{2\pi}\right)^{1/2} \sum_n e^{ikan}\chi(q - na). \tag{7}$$

It is easy to check that the function $\chi(X)$ in Eq. (5) leads to $|C^{(d)}(kq)| = \text{const}$. A normalized kq function that satisfies Eq. (6) is necessarily a pure phase factor

$$C(k,q) = \frac{1}{\sqrt{2\pi}}\exp[i\varphi(k,q)], \tag{8}$$

where $\varphi(k,q)$ is real. In the X representation the condition given by Eq. (6) [or Eq. (8)] assumes the following form [Eq. (7) is used]:

$$a\sum_n \chi^*(X - na)\chi(X - na - \ell a) = \delta_{\ell 0}. \tag{9}$$

Again it is easy to check that the function in Eq. (5) with $d = a$ satisfies Eq. (9).

This is as far as the Balian-Low theorem goes. Now we shall connect it with the dynamics of an electron in a magnetic field B. When the motion is in the xy plane with the magnetic field $\vec{B}$ in the z direction, Schrödinger's equation assumes the form [the Landau gauge $\vec{A} = (0, Bx)$ is chosen]

$$\left[\frac{p_x^2}{2m} + \frac{[p_y + (\hbar/\lambda^2)x]^2}{2m}\right]\psi(x,y) = E\psi(x,y), \qquad \lambda^2 = \frac{\hbar c}{eB}. \tag{10}$$

Equation (10) is written in the xy representation. For the magnetic field problem it is convenient to work in the WX representation according to the transformation

$$W = \frac{m\lambda^2}{\hbar}v_y = \frac{\lambda^2}{\hbar}\left(p_y + \frac{\hbar}{\lambda^2}x\right), \qquad P_W \equiv mv_x = p_x,$$
$$X = \frac{\lambda^2}{\hbar}\left(p_x + \frac{\hbar}{\lambda^2}y\right), \qquad P_X = p_y = \frac{\hbar}{\lambda^2}Y. \tag{11}$$

Here WP_W denotes the velocity degree of freedom and XP_X the orbit center degree of freedom (as was already used above). The relation between the wave function $\psi(x,y)$ in the xy representation and the wave function $\phi(WX)$ in the

WX representation is [17]

$$\psi(x,y) = \frac{1}{2\pi\lambda^2}\int\int \exp\left[-\frac{i}{\lambda^2}(xy + WX - xX - yW)\right]\phi(W,X)\,dW\,dX\,. \qquad (12)$$

From Eqs. (10) and (11) it follows that the Hamiltonian depends on the degree of freedom WP_W only (it does not depend on the orbit center). The wave function $\phi(W,X)$ can therefore be chosen as a product function

$$\phi_\ell(W,X) = \gamma_\ell(W)\chi(X)\,, \qquad (13)$$

where ℓ labels the Landau levels. On the other hand, the magnetic translations depend on the orbit center operators X and P_X only [9,12]. In particular, the commuting finite magnetic translations in x and y directions can be written in the following way [6]

$$\tau_x(d) = \exp\left[\frac{i}{\hbar}\left(p_x + \frac{\hbar}{\lambda^2}y\right)Nd\right] \equiv \exp\left(iX\frac{2\pi}{d}\right),$$
$$\tau_y(Nd) = \exp\left(\frac{i}{\hbar}p_y d\right) \equiv \exp\left(\frac{i}{\hbar}P_X d\right), \qquad (14)$$

where N is an integer which is defined by the following rationality condition [9]

$$N = \frac{hc/e}{Bd^2} = \frac{2\pi\lambda^2}{d^2}\,. \qquad (15)$$

This relation has a simple meaning of the ratio of the elementary fluxon hc/e to the flux of the magnetic field B through a unit cell of area d^2 [d is an arbitrary constant [18], see Eq. (4)]. As is seen from Eq. (14), the magnetic translations depend only on the XP_X degree of freedom of the orbit center. We can therefore use directly the results in Eqs. (1)–(4) for constructing a von Neumann set for a Landau level, by replacing $\chi(X)$ in Eq. (13) by $\chi_{mn}(X)$ in Eq. (4). We get

$$\phi_{\ell mn}(W,X) = \gamma_\ell(W)\chi_{mn}(X)\,. \qquad (16)$$

Now, when $\chi(X)$ satisfies the condition given by Eq. (6) or (9), the set of functions in Eq. (16) for each Landau level ℓ is orthogonal. By using the transformation in Eq. (12) we get the orthonormal von Neumann set $\psi_{\ell mn}(x,y)$ in the xy representation [the operators in Eq. (11) are used]

$$\psi_{\ell mn}(x,y) = (-1)^{mn}\exp\left(\frac{i}{\lambda^2}yN\,dn\right) \times \psi_\ell(x + N\,dn, y + dm)\,. \qquad (17)$$

The function $\psi_\ell(x,y)$ in Eq. (17) is found according to Eqs. (12) and (13), and can be given the two alternative forms

$$\psi_\ell(x,y) = \frac{1}{\sqrt{2\pi}}\int \exp(iyk)\gamma_\ell(k\lambda^2 + x)F_\chi(k)\,dk$$
$$= \frac{\exp(-ixy/\lambda^2)}{\lambda^2\sqrt{2\pi}} \times \int \exp\left(\frac{i}{\lambda^2}xz\right)F_\gamma\left(\frac{z-y}{\lambda^2}\right)\chi(z)\,dz\,, \qquad (18)$$

where F_χ and F_γ are the Fourier transforms of χ and γ, respectively [γ and χ are the functions in Eq. (13)]. It should be pointed out that the condition given by Eq. (6) [or Eq. (9)] is necessary and sufficient for the von Neumann set [Eqs. (16) or (17)] for each Landau level ℓ to be orthogonal.

By using the results for the single degree of freedom XP_X $[P_X = (\hbar/\lambda^2)Y]$, the Balian-Low theorem implies that the coordinates X and Y of the orbit center in the state ψ_ℓ [Eq. (18)] for a magnetic field cannot both be well localized, or in a more precise language, at least one of the two uncertainties ΔX and ΔY in the ψ_ℓ state [Eq. (18)] diverges. It is interesting to point out that for obtaining the result there is no need to go to the configuration plane xy [see Eq. (12)]. However, since the coordinates x and y of the electron in a magnetic field are related to the operators WP_W and XP_X $[P_X = (\hbar/\lambda^2)Y]$ in Eq. (11), one has

$$x = W - Y, \qquad y = X - \frac{\lambda^2}{\hbar}P_W\,. \qquad (19)$$

Bearing in mind that the uncertainties ΔW and ΔP_W are finite in the state given by Eq. (13) [or alternatively, by Eq. (18)], we come to the conclusion [by using Eq. (18)] that at least one of the uncertainties Δx or Δy diverges in the state $\psi_\ell(x,y)$ of Eq. (18). This also means that the electron in a magnetic field cannot be well localized in both the x and y directions.

It is of interest to compare the consequences of the Balian-Low theorem with the known results for the Wannier function in a magnetic field [8,19]. In Ref. [8] it was shown that Wannier functions for a Bloch electron in the xy plane, with the magnetic field B perpendicular to the plane, cannot fall off at infinity faster than r^{-2} $(r^2 = x^2 + y^2)$. In Ref. [19] Wannier functions with a r^{-2} falloff were actually constructed for such a two-dimensional problem where only a magnetic field is present. One can see that the Balian-Low theorem for Landau levels is in good agreement with the results of Refs. [8] and [19]. In order to see it, we notice that as a consequence of the Balian-Low theorem the quantity $\langle x^2 + y^2\rangle = \langle r^2\rangle$ diverges (the angular brackets denote the expectation value), because either $\langle x^2\rangle$ or $\langle y^2\rangle$ has to diverge. This divergence of $\langle r^2\rangle$ also follows when the function falls off as r^{-2} at infinity, like in Ref. [19].

It should, however, be pointed out that the Balian-Low theorem is completely general in nature. For example, when $\chi(X)$ is given by Eq. (5), then $\psi_0(x,y)$ in Eq. (18) assumes the form (for the lowest Landau level, $\ell = 0$)

$$\psi_0(x,y) = \left(\frac{1}{4\pi^3 d^2\lambda^6}\right)^{1/4}\exp\left(-\frac{ixy}{\lambda^2}\right) \times \int_{-d/2}^{d/2}\exp\left[\frac{i}{\lambda^2}xy - \frac{(y-z)^2}{2\lambda^2}\right]dz\,. \qquad (20)$$

In deriving Eq. (20), use was made of the second line in Eq. (18). For large y, Eq. (20) becomes

$$\psi_0(x,y) \sim \left(\frac{d^2}{4\pi^3\lambda^6}\right)^{1/4} \exp\left(\frac{ixy}{\lambda^2} - \frac{y^2}{2\lambda^2}\right) \times \sin\left(\frac{xd}{2\lambda^2}\right) \Bigg/ \left(\frac{xd}{2\lambda^2}\right). \tag{21}$$

This function has a $\frac{1}{x}$ falloff in the x direction and is Gaussian in the y direction.

Finally, it can be directly checked that the von Neumann set in Eq. (17) is orthonormal, when $\psi_\ell(x,y)$ is given by Eq. (18). In the particular case of the localized function in Eq. (20), the explicit orthonormal von Neumann set is [see Eq. (17)]

$$\psi_{0mn}(x,y) = \frac{(-1)^{mn}}{(4\pi^3 d^2\lambda^6)^{1/4}} \exp\left(-\frac{i}{\lambda^2}xy - \frac{i}{\lambda^2}xmd\right) \int_{-d/2}^{d/2} \exp\left[\frac{i}{\lambda^2}(x + nNd)z - \frac{(y + md - z)^2}{2\lambda^2}\right] dz\,. \tag{22}$$

In conclusion, the Balian-Low theorem was applied to the motion of an electron in the xy plane, with a magnetic field perpendicular to this plane. By using this theorem it was shown that orthonormality on a von Neumann lattice and localizability are incompatible features. In particular, it is shown that even for the best localized eigenfunctions for the Landau levels the uncertainties Δx and Δy can never both be made finite. On a qualitative level, incompatibility of orthonormality and localizability is not an unexpected feature for wave functions in quantum mechanics [20]. In this Letter we derive for the first time a precise quantitative result for Landau level wave functions in a magnetic field. Although the proof was carried out in the Landau gauge, the result that Δx and Δy can never both be finite in a Landau level state $\psi_\ell(x,y)$ [see Eq. (17)], for any ℓ, is gauge independent. This is seen from the expression of the wave function in the WX representation [Eq. (16)]. The part of the wave function $\chi_{mn}(X)$ in Eq. (16) that leads to the von Neumann set does not depend on the gauge because the X coordinate does not appear in the Hamiltonian of the problem [Eq. (10)].

As is well known, in the symmetric gauge the wave function for any Landau level can be chosen well localized in both x and y directions [13]. However, for such a well localized wave function, the von Neumann set in Eq. (17) will *not be orthogonal*.

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