Backflow in a Fermi Liquid

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We calculate the backflow current around a fixed impurity in a Fermi liquid. The leading contribution at long distances is radial and proportional to $1/r^2$. It is caused by the current induced density modulation first discussed by Landauer [IBM J. Res. Dev. **1**, 223 (1957)]. The familiar $1/r^3$ dipolar backflow obtained in linear response is only the next-to-leading term, whose strength is calculated here to all orders in the scattering. In the charged case the condition of perfect screening gives rise to a novel sum rule for the phase shifts. Similar to the behavior in a classical viscous liquid, the friction force is due only to the leading contribution in the backflow while the dipolar term does not contribute. [S0031-9007(97)04866-7]

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The calculation of backflow in liquids is one of the standard problems in hydrodynamics, determining, e.g., the Stokes friction in a classical viscous liquid [1] or the properties of rotons and impurities in superfluid Helium 4 [2]. For the case of a Fermi liquid, the backflow around a slowly moving massive impurity is discussed in the classic text by Pines and Nozieres [3]. Within linear response they show that the backflow is proportional to the density response function and dipolar in character. As pointed out by these authors, the dipolar form may be derived from a rather simple geometrical argument: Indeed, the backflow current outside the impurity should have zero divergence being a stationary flow and zero curl because the perturbation is longitudinal. The only vector function obeying both conditions, however, is a dipole. For a neutral Fermi liquid the strength of the dipole is given by the compressibility times the Fourier transform of the interaction at zero momentum. In the charged case the dipolar backflow has a universal amplitude. This is a result of perfect screening which requires that the backflow identically cancels the longitudinal part of the impurity current [3].

In this Letter we reconsider the backflow problem in a Fermi liquid, going beyond the linear response treatment. Starting with the simple case of a noninteracting Fermi gas, we show that the leading term at long distances is not the dipolar backflow but a radial contribution decaying like $1/r^{d-1}$ in d dimensions ($d = 2, 3$ in the following). It is proportional to the impurities transport cross section and thus is not contained in a linear response calculation where the interaction only appears linearly. The novel term has nonzero curl and is directly related to the asymmetry in density around localized scatterers in the presence of a finite transport current, discussed long ago by Landauer [4]. The result is easily generalized to interacting Fermi liquids, where it applies to the low temperature, collisionless regime. We also calculate the familiar dipolar backflow to all orders in the scattering potential. It is found that the condition of perfect screening entails a sum rule for the scattering phase shifts which is similar to, but different from the one, by Friedel [5]. Finally, we determine the systematic force exerted on the impurity by the moving fermions. It is shown that only the leading $1/r^{d-1}$ term of the backflow current contributes to the force, a situation which is completely analogous to that in a classical viscous liquid.

Let us consider a fixed scattering center at the origin which is characterized by a spherically symmetric interaction potential $V(\vec{x})$. In the frame where the impurity is at rest, the Fermi system is flowing past with asymptotic velocity $\vec{v} \neq 0$. The unperturbed current density is therefore $\vec{j}(\vec{x})|_0 = n\vec{v}$, with n the equilibrium number density. Because of scattering off the impurity, the actual current density $\vec{j}(\vec{x})$ differs from $n\vec{v}$ by a backflow current $\delta\vec{j}(\vec{x})$. To lowest order in $\vec{v}$ the Fourier transform $\delta\vec{j}(\vec{q})$ of the backflow is of the form

$$\delta\vec{j}(\vec{q}) = h(q)[(\hat{q} \cdot \vec{v})\hat{q} - \vec{v}], \tag{1}$$

where $\hat{q}$ is the unit vector in the direction of $\vec{q}$. Indeed, the vector in Eq. (1) is uniquely determined by the requirement that it is linear in $\vec{v}$ and the zero divergence condition $\vec{q} \cdot \delta\vec{j}(\vec{q}) = 0$ due to the stationarity of the flow. For small velocities the backflow pattern is thus completely determined by the scalar function $h(q)$. As pointed out above, a treatment of the interaction potential $V(\vec{x})$ in linear response gives rise to a dipolar backflow which is characterized by $\lim_{q\to 0} h(q) = h_0$. The associated dimensionless constant h_0 is equal to $\partial n/\partial\mu \cdot V(q = 0)$ in the case of a neutral Fermi liquid [3]. Here, the compressibility $\partial n/\partial\mu$ is just the $q \to 0$ limit of the general density response function $\chi(q)$. For an impurity with charge Z, the product $\chi(q)V(q)$ is replaced by $Z[\epsilon^{-1}(q) - 1]$ with $\epsilon(q)$ the static dielectric constant [3]. As a result of the perfect screening condition $\epsilon^{-1}(q \to 0) = 0$, this leads to a universal value $h_0^c = -Z$ for the strength of the dipolar backflow in the charged case.

In order to discuss the generalization of these results beyond linear response, still, however, keeping the asymptotic velocity $\vec{v}$ small, we start by considering a noninteracting Fermi gas. In this case, the backflow can be calculated analytically from the single particle eigenstates

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$\psi_k(\vec{x})$, which are the exact outgoing scattering states in $V(\vec{x})$. Indeed, describing the finite asymptotic current $n\vec{v}$ by a shifted Fermi distribution $f(\epsilon_{\vec{k}-m\vec{v}/\hbar})$ for the incoming momenta $\vec{k}$, the total fermionic current density at zero temperature and to linear order in $\vec{v}$ is given by

$$\vec{j}(\vec{x}) = \frac{k_F^{d-1}}{(2\pi)^d} \int d\Omega_k \hat{k} \cdot \vec{v}\, \mathrm{Im}[\psi_k^*(\vec{x}) \nabla_x \psi_k(\vec{x})]|_{k=k_F} . \qquad (2)$$

Here $d\Omega_k$ denotes an integration over the directions of the unit vector $\hat{k}$, while the magnitude $k = |\vec{k}|$ is fixed at the Fermi wave vector k_F. Thus, at $T = 0$ and to linear order in $\vec{v}$, the backflow is completely determined by the exact scattering states right at the Fermi surface. Clearly, the behavior of $\vec{j}(\vec{x})$ at arbitrary distances depends on the details of $\psi_k(\vec{x})$. For large distances, however, it is sufficient to know the asymptotic form of the scattering states. In order to obtain the first two leading contributions to $\vec{j}(\vec{x})$, it is necessary to expand

$$\psi_k(\vec{x}) \to e^{i\vec{k}\cdot\vec{x}} + f \frac{e^{ikr}}{r} + f_2 \frac{e^{ikr}}{r^2} + \ldots \qquad (3)$$

to order $1/r^2$ in three dimensions. Here, f is the standard scattering amplitude while the coefficient of the $1/r^2$ contribution is given by

$$f_2 = \frac{i}{2k^2} \sum_{l=0}^{\infty} (2l + 1) l(l + 1) e^{i\delta_l} \sin \delta_l P_l(\hat{k} \cdot \hat{x}) \qquad (4)$$

with phase shifts δ_l and the usual Legendre polynomials P_l. This result is obtained by a straightforward asymptotic expansion of the free particle solutions with given angular momentum l. In two dimensions the corresponding result turns out to be

$$\psi_k(\vec{x}) \to e^{i\vec{k}\cdot\vec{x}} + f \frac{e^{ikr}}{r^{1/2}} + f_2 \frac{e^{ikr}}{r^{3/2}} + \ldots \qquad (5)$$

with amplitudes f and f_2 which are not given here explicitly. It is now straightforward to insert the asymptotic behavior of the scattering states into our expression (2) for the current. Apart from the trivial term $\vec{k}$, which accounts for the background current density $n\vec{v}$, the leading contributions to $\mathrm{Im}[\psi^* \nabla \psi]|_{k=k_F}$ obviously arise from the square $k_F \hat{x} |f|^2 / r^{d-1}$ of the outgoing wave and the two interference terms linear in f. Now, $\exp i(kr - \vec{k} \cdot \vec{x})$ is asymptotically proportional to a δ function, $\delta(\Omega_k - \Omega_x)$, which singles out the forward direction $\hat{k} = \hat{x}$. Using the optical theorem, it is straightforward to show that the leading term to the backflow is given by [6]

$$\delta\vec{j}(\vec{x}) \to -\frac{k_F^d}{(2\pi)^d} \frac{\sigma_{\mathrm{tr}}}{r^{d-1}} (\hat{x} \cdot \vec{v})\hat{x} + O(r^{-d}) \qquad (6)$$

with $\sigma_{\mathrm{tr}} = \int d\Omega_k (1 - \hat{k}\hat{x}) |f|^2$ the standard transport cross section. Obviously, the contribution (6) is a purely radial current which vanishes in the direction perpendicular to $\vec{v}$ (see Fig. 1). It has vanishing divergence as it should, but finite curl.

In order to understand its physical origin we consider the current induced part $\delta n(\vec{x})$ of the density modulation which is caused by the scattering off the impurity. As predicted by Landauer [4] and recently verified experimentally [7], this modulation asymptotically has the form $\delta n(\vec{x}) \sim -(\hat{x} \cdot \vec{v})/r^{d-1}$ of a dipole potential. Comparing the exact expression obtained for $\delta n(\vec{x})$ in a scattering theory calculation [8] with our result (6), it turns out that, at $T = 0$ and to linear order in $\vec{v}$, the asymptotic backflow current is simply given by

$$\delta\vec{j}(\vec{x}) = v_F \delta n(\vec{x}) \cdot \hat{x} . \qquad (7)$$

The leading term in the backflow is thus directly proportional to the current induced density change $\delta n(\vec{x})$ which is positive in front and negative behind the scatterer, in agreement with the intuitive picture developed by Landauer [4]. As a result, the sign of this contribution to the backflow remains unchanged upon going from a repulsive to an attractive potential $V(\vec{x})$. This is in contrast to the

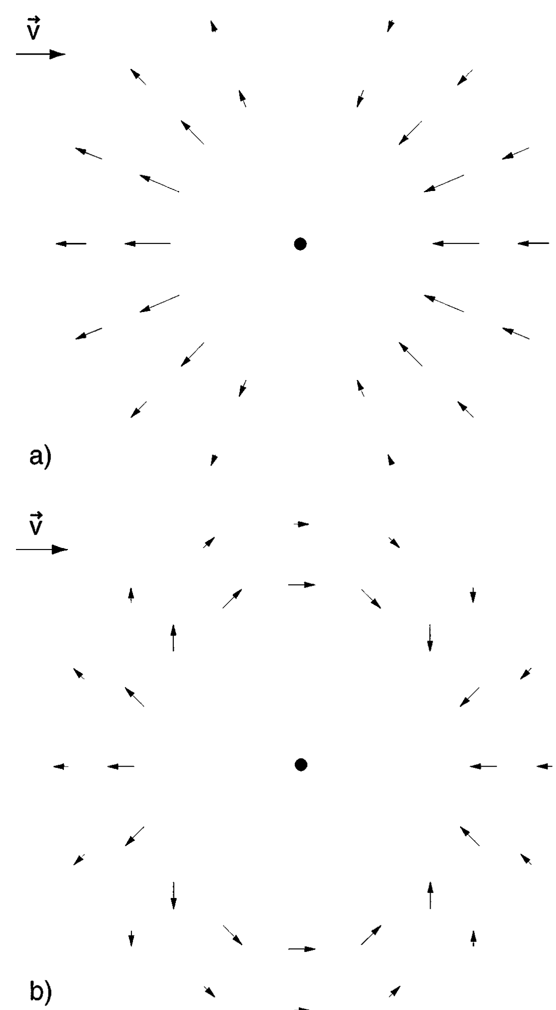

FIG. 1. (a) Radial backflow current $-(\hat{x} \cdot \vec{v})\hat{x}/r$ in $d = 2$. The impurity sits at the center with incoming current from the left. (b) Dipolar backflow $[\vec{v} - 2(\hat{x} \cdot \vec{v})\hat{x}]/r^2$ in $d = 2$ for a repulsive impurity at the center. The direction of the flow is reversed in the attractive case, in contrast to (a).

dipolar contribution,

$$\delta\vec{j}(\vec{x})|_{\rm dip} = -\frac{h_0}{2\pi(d-1)}\frac{d(\hat{x}\cdot\vec{v})\hat{x}-\vec{v}}{r^d}, \qquad (8)$$

which is only the next-to-leading term in an asymptotic expansion of $\vec{j}(\vec{x})$. Including the subleading amplitude f_2 in (3) and (5), a straightforward but rather tedious calculation indeed gives a contribution to $\delta\vec{j}(\vec{x})$ of the form (8) with strength

$$h_0 = \frac{\partial n}{\partial \mu}\,\mathrm{Re}\,V_0 - \frac{2}{\pi} \times \sum_{l=0}^{\infty} c_l \sin\delta_l \sin\delta_{l+1}\sin(\delta_l - \delta_{l+1}). \qquad (9)$$

Here $c_l = 2l + 1$ or $(l+1)^2$ in $d = 2$ or $d = 3$, respectively, while

$$V_0 = \int d^d x\, e^{-i\vec{k}\cdot\vec{x}} V(\vec{x})\psi_k(\vec{x}) \qquad (10)$$

is essentially the exact forward scattering amplitude. Since V_0 reduces to $V(q = 0)$ in Born approximation, the first term in (9) is the obvious generalization of the linear response result of Pines and Nozieres [3] to arbitrary order in the scattering potential. It is convenient to also express this contribution in terms of the phase shifts δ_l via

$$h_0^{(1)} = -\frac{1}{2\pi}\sum_{l=0}^{\infty} a_l \sin 2\delta_l, \qquad (11)$$

where $a_l = 2 - \delta_{l,0}$ or $2l + 1$ in $d = 2, 3$. In addition to $h_0^{(1)}$ there is a second contribution which is at least of order V^3. It arises from the interference term $(k_F\hat{x}/r^d)\,(2\,\mathrm{Re}\,f^* f_2)$ between the first and next-to-leading contribution to the outgoing wave. The additional term is odd under $\delta_l \to -\delta_l$ as is the first one, but vanishes in the case of s-wave scattering only.

Including both the radial and dipolar contributions to the backflow current, the scalar function $h(q)$ defined in (1) has the general form

$$\lim_{q\to 0} h(q) = \frac{h_{-1}}{q} + h_0 + \ldots \qquad (12)$$

with $h_{-1} = (2, 3\pi/4)n\sigma_{\rm tr}$ in $d = (2, 3)$. Since

$$\sigma_{\rm tr} = \frac{4}{k_F^{d-1}}\sum_{l=0}^{\infty} b_l \sin^2(\delta_l - \delta_{l+1}) \qquad (13)$$

with $b_l = 1$ or $\pi(l + 1)$ in $d = 2, 3$, both leading coefficients h_{-1} and h_0 can be expressed completely in terms of the density $n \sim k_F^d$ and the scattering phase shifts δ_l. In the particular case of $d = 3$ and s-wave scattering with scattering length a, we have $h_{-1} = (k_F a)^2 k_F/2$ and $h_0 = k_F a/\pi$. With k_F as the typical scale for q, this shows that the strength of the leading radial term is then a factor $k_F a \ll 1$ smaller than the dipolar contribution. Nevertheless, at long distances it is always the radial term which dominates.

In a Fermi liquid the interacting state develops adiabatically from the noninteracting one. The resulting quasiparticle states in a local and centrally symmetric potential are characterized by phase shifts δ_l. In general, these are functionals of both the energy and the quasiparticle distribution $n_k(\vec{x})$ [9]. By Galileian invariance the asymptotic distribution is again a Fermi sphere shifted by $\delta\vec{k} = m\vec{v}/\hbar$, with m the bare mass. At $T = 0$ the energy is fixed at ϵ_F, and there are no collisions other than with the impurity. Moreover, since the deviation of n_k from equilibrium is already linear in $\vec{v}$, we may neglect the dependence of δ_l both on energy and on n_k. The resulting values of δ_l then define an effective force $\vec{F}_k$ on the quasiparticles which appears in the corresponding transport equation [3]. In a fully quantum mechanical treatment of the Wigner function $n_k(\vec{x})$, the associated local particle current must then be equal to $\vec{j}_k(\vec{x}) = \frac{\hbar}{m}\,\mathrm{Im}\,\psi_k^*\nabla_x\psi_k$, where ψ_k are the exact scattering states in an effective potential with phase shifts δ_l. As was shown above, h_{-1} and h_0 can be expressed completely in terms of k_F and the scattering phase shifts δ_l. The generalization of our results to the interacting case is therefore rather obvious, provided that the collision term in the transport equation is irrelevant. Indeed, since k_F is unchanged, one only needs to replace the phase shifts by those for quasiparticles. The general form of the backflow as determined by (1) and (12) thus applies also in the interacting case, however, with renormalized parameters h_{-1} and h_0. For a charged impurity in an electron liquid, the perfect screening condition must hold to all orders in V. As we have seen, this implies a universal dipolar backflow characterized by $h_0^c = -Z$ for an impurity with charge Z. Since h_0 is completely determined by the δ_l via (9) and (11), perfect screening gives a nontrivial condition on the scattering phase shifts at a charged impurity. In the limit $\delta_l \ll 1$ it reduces to the well-known Friedel sum rule [5] which fixes the number of bound states. The novel sum rule shows that even for $Z = 1$ no purely s-wave scattering potential can account for the backflow in the charged case. Regarding the dominant radial contribution, the transport cross section appearing in the coefficient h_{-1} has to be replaced by its value for the screened potential $V(q)/\epsilon(q)$. In contrast to h_0, the strength of the radial backflow is therefore not universal.

Our discussion up to now has been restricted to the zero temperature limit, where only properties right at the Fermi energy are relevant. For the noninteracting problem, the generalization to finite temperatures is trivial. Indeed, in this case, the basic result (12) remains valid at arbitrary temperatures; however, the coefficients h_{-1} and h_0 are averaged over energy with the negative derivative of the Fermi distribution. In the interacting case the situation is more complicated since, for nonzero temperatures, the quasiparticles have a finite mean free path $\ell = v_F\tau \sim T^{-2}$ leading to a nonvanishing kinematic viscosity $\nu \approx v_F\ell$ of the liquid. The local particle current can therefore be obtained from effective single particle eigenstates only if scattering between quasiparticles is negligible. In a Fermi liquid, our results are thus valid only in the low temperature, collisionless regime $\ell \gg R$, where R is the size of the impurity. At higher temperatures, one eventually enters the hydrodynamic regime $\ell \ll R$, where collisions between the fermions play a crucial role. At present, there seems to be no solution of the backflow problem in this regime, except in the extreme limit of a classical and incompressible liquid with kinematic viscosity ν. In this case, the problem may be treated by using the linearized Navier-Stokes equation with boundary condition $\vec{v} = 0$ at the surface of the impurity. Taking a sphere of radius R, one finds [1] that $\delta\vec{j}(\vec{x})$ has a contribution proportional to $1/r$ and a dipolar one. The associated function $h(q)$ as defined in (1) is thus of the form

$$\lim_{q\to 0} h_{\rm cl}(q) = \frac{h_{-2}}{q^2} + h_0 + \ldots . \tag{14}$$

The coefficient of the $1/r$ contribution is $h_{-2} = 6\pi R n$ while the strength of the dipolar backflow is negative and given by $h_0 = -\pi R^3 n$ (the corresponding problem in two dimensions has no solution, which is known as the Stokes paradox). In a classical viscous liquid, the $1/r^2$ backflow found for a Fermi liquid at low temperatures is thus absent and replaced by a $1/r$ contribution. It is an interesting open problem to study how the backflow changes from the quantum result (12) to the quite different expression (14) for the classical, incompressible case.

Finally, we calculate the systematic force $\vec{F}$ due to the transfer of momentum between liquid and scatterer. Taking the gradient of the interaction energy with respect to the impurity position, it is straightforward to see that

$$\vec{F} = -\int d^d x\, n(\vec{x}) \nabla_x V . \tag{15}$$

Clearly, at zero current $\vec{v} = 0$, this force vanishes although the fermion density is not uniform even in this case. Therefore only the current induced density change $\delta n(\vec{x})$ contributes to $\vec{F}$. For simplicity we consider again a Fermi gas at $T = 0$ with scattering states $|\vec{k}+\rangle$. To lowest order in $\vec{v}$, the force can then be written as

$$\vec{F} = \frac{k_F^{d-1}}{(2\pi)^d} \int d\Omega_k\, \hat{k} \cdot \vec{v}\, \frac{m}{\hbar} \langle \vec{k} + | - \nabla_x V | \vec{k}+\rangle|_{k=k_F} \tag{16}$$

similar to (2). Now the relevant matrix element of $\nabla_x V$ between the exact scattering states is equal to $2\epsilon_F \sigma_{\rm tr}(k_F) \cdot \hat{k}$. Thus (16) immediately gives a conventional friction force $\vec{F} = -\eta_F \vec{v}$ with $\eta_F = \hbar k_F n \sigma_{\rm tr}$ [6]. The fermionic friction coefficient η_F is proportional to the transport cross section which appears in the radial contribution h_{-1} to the backflow. It is this term which determines the single impurity contribution to the residual resistivity [4,8]. This is a simple example of the so-called Das-Peierls theorem [10] in electromigration, which states that the total force on the impurity is proportional to the additional resistivity is causes. The fact that the dipolar contribution h_0 to the backflow does not contribute to the friction force can be understood most easily by considering the linear response regime. Indeed, to linear order in V the response at low velocities is purely reactive [3], while a finite resistivity can only appear at order V^2. More generally, the coefficient h_0 is odd in δ_l, while the force must be an even function of the phase shifts. This situation is, in fact, very similar to the case of a classical, incompressible, and viscous liquid. Calculating the frictional force $\vec{F} = -\eta_S \vec{v}$ in a fluid with kinematic viscosity ν which is associated with the corresponding backflow pattern (14), it turns out [1] that only the leading term h_{-2} contributes to $\eta_S = 6\pi R n m \nu$ while the dipolar backflow again drops out. Comparing the Stokes result with that for a Fermi liquid, we see that the fermionic friction coefficient for a scattering potential with characteristic range R such that $\sigma_{\rm tr} = \pi R^2$ is equal to that of a classical liquid with finite kinematic viscosity $\nu_F = v_F R/6$. With typical values $R = 2$ Å and $v_F = 1.5 \times 10^8$ cm/s for electrons in metals, we obtain $\nu_F = 0.5$ cm^2/s which is about fifty times the viscosity of water. From this point of view, therefore, electrons in metals behave like a rather viscous liquid.

In summary, we have calculated the backflow and the induced force due to a fixed impurity in a Fermi liquid at low velocities and temperature. Similar to the related concept of the Landauer resistivity dipole, our calculation provides a microscopic understanding of the basic phenomenon of residual resistance. It would therefore be very interesting if scanning microscopy, which has been successfully used [7] to detect the Landauer dipole, could also monitor local current distributions. Recent progress in this direction [11] shows that this may be possible in the near future.

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