Synchronizing Spatiotemporal Chaos of Partial Differential Equations

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A general approach for synchronizing pairs of unidirectionally coupled partial differential equations (PDEs) with spatiotemporally chaotic dynamics is introduced. We show that for a large class of PDEs, a pair of PDEs can be synchronized by driving the response system only at a *finite* number of space points. We also discuss the relevance of our results for control of spatiotemporal chaos. [S0031-9007(97)03498-4]

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Chaos synchronization has recently aroused a great deal of interest in the light of potential applications in engineering [1]. Techniques based on the Pecora-Carroll [2] method have been very successful for synchronizing chaos in low-dimensional systems [3,4]. Synchronizing spatiotemporal systems remains a challenge, however, because the chaotic states in such systems are typically high dimensional, involving multiple stable and unstable modes. Recently it was shown how to synchronize hyperchaotic systems with a scalar continuous signal [4,5], while in [6] a method for synchronization of spatiotemporal chaos of two arrays of coupled nonlinear oscillators is discussed. In this Letter we describe a general method for synchronizing *pairs* of unidirectionally coupled partial differential equations (PDEs) with spatiotemporal chaotic dynamics. We mention a few examples where the study of interaction between spatially extended systems is important: reentry initiation in coupled parallel fibers [7], dynamics of multilayered natural and artificial neural networks [8], thermal convention in multilayered media [9], and systems which consist of several spatially extended systems that are weakly coupled, an example being the electrohydrodynamical convention in liquid crystals [10].

The Letter is organized as follows. First we illustrate numerically how the method leads to synchronization of spatiotemporal chaos [11] of pairs of Gray-Scott equations [12]. Then we give arguments why it can be expected that the coupling mechanism used leads to synchronization for a large class of pairs of PDEs, and finally we discuss the relevance of our results for control of chaos.

To demonstrate spatiotemporal synchronization of PDEs, we use as an example the Gray-Scott cubic autocatalysis model to simulate a 1D reaction-diffusion system exhibiting mixed-mode spatiotemporal chaos [13],

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= -u_1 v_1^2 + a(1 - u_1) + D_u \nabla^2 u_1 \,, \\ \frac{\partial v_1}{\partial t} &= u_1 v_1^2 - (a + b) v_1 + D_v \nabla^2 v_1 \,, \end{aligned} \tag{1}$$

where u_1 and v_1 represent the substrate and activator concentration, respectively, a and b are parameters of the reaction, and D_u and D_v are the diffusion constants. Let L be the linear extension of the reactor tank, and $N = [L/X]$ be the number of equidistant points. Equation (1) drives a similar PDE,

$$\begin{aligned} \frac{\partial u_2}{\partial t} &= -u_2 v_2^2 + a(1 - u_2) + D_u \nabla^2 u_2 \,, \\ \frac{\partial v_2}{\partial t} &= u_2 v_2^2 - (a + b) v_2 + D_v \nabla^2 v_2 + f(x,t) \,. \end{aligned} \tag{2}$$

Let $T > 0$ and $X > 0$ be real numbers. Let $v_2(t - 0)$ be the value of the signal v_2 immediately prior to the time t. The driving function $f(x,t)$ influences the response system in the following way: at each moment $t = kT$ ($k \in Z$), N space points $x = 0, X, 2X, \ldots, (N - 1)X$ are simultaneously driven and their corresponding v_2 variables are set to new values $v_2(kT) = v_2(kT - 0) + \varepsilon[v_1(kT) - v_2(kT - 0)]$. During the rest of the time $t \neq kT$ PDEs (1) and (2) are not connected and oscillate independently from each other. Thus, T denotes the time distance between the occurrence of the driving impulses, and X is the space distance between the driven space points. Note that in the case when $X = T = 0$ and $\varepsilon = 1$ this driving method becomes the Pecora-Carroll approach for synchronization in PDEs. The motivation for such a driving as in Eq. (2) is twofold: (i) to enable the synchronization where only a *finite* number of space points are controlled; (ii) to make possible the synchronization through *time-discontinuous* monitoring and influencing the state variables.

The results of our numerical simulations are shown in Fig. 1. We have used an explicit Euler scheme and periodic boundary conditions with $M = 256$ mesh points in space, and a time integration step size of $\Delta t = 0.05$.

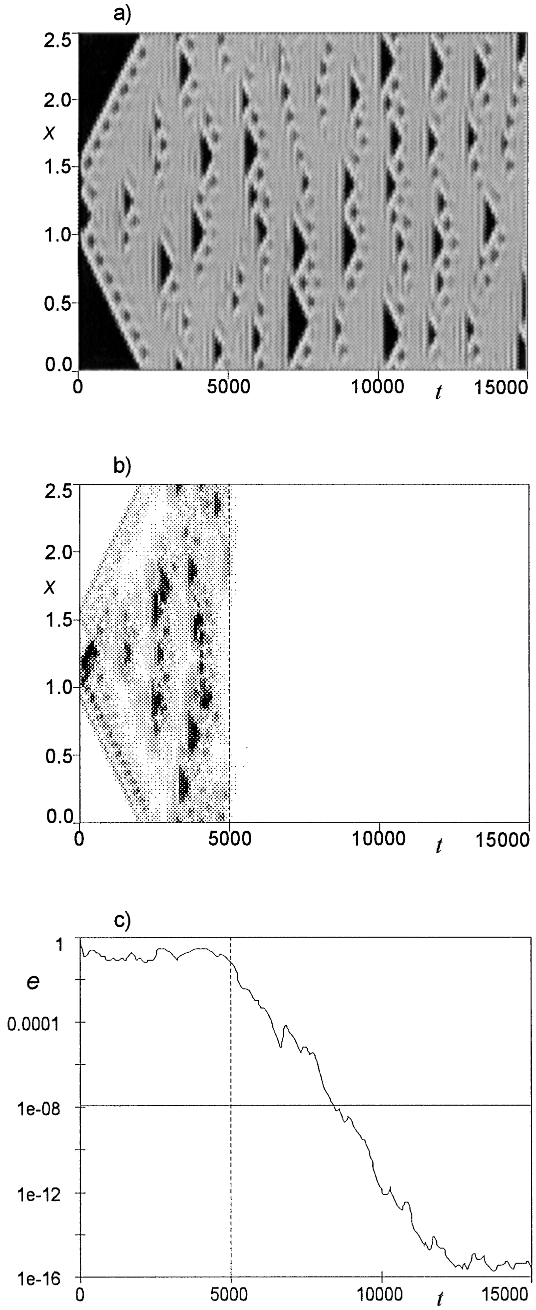

FIG. 1. Synchronization of spatiotemporal chaos. (a) Spatiotemporal evolution of the u variable of the PDE (1) as a function of time t and spatial coordinate x. (b) The same as in (a) for the difference $|u_1 - u_2|$. The dashed line denotes the time $t = 5000$ when the coupling between the PDEs is switched on. (c) The synchronization error Eq. (3) versus time.

The values of the parameters are [14] $a = 0.028$, $b = 0.053$, $D_v = 1.0 \times 10^{-5}$, $D_u = 2D_v$, and $L = 2.5$. The system is initiated with $u(x) = 1$ and $v(x) = 0$ with a strong perturbation of the center region. Figure 1(a) depicts the spatiotemporal evolution of the PDEs (1).

In the simulation we have used $\varepsilon = 0.2$, $T = 20\Delta t$, and $X = (8/256)L = 0.031\,25L$. In other words, $N = 32$ space points are driven. This number remains unchanged even if M is increased to $M = 512$ and $M = 1024$. Moreover, we find that there exists a critical value X_{cr} such that for all $X < X_{\mathrm{cr}}$ both systems (1) and (2) are synchronized. For the above parameter values, $X_{\mathrm{cr}} = (14/256)L$, and this number also remains unchanged with increasing M. This is a remarkable result: it shows that two infinite dimensional systems can be synchronized using a finite number of coupling points, or, in other words, the synchronization can be achieved applying an N dimensional vector as a driving signal.

In order to visualize the synchronization, we switch on the coupling at $t = 5000$ as denoted by the dashed line. Figure 1(b) shows the modulus of the difference of the u variables of the drive and the response system. The dark regions indicate the desynchronization of the PDEs, in particular, during the time interval $0 < t < 5000$ when the coupling is switched off. This effect can also be seen in Fig. 1(c) that gives the global synchronization error e

$$e = \sqrt{\frac{1}{L}\int_0^L [(u_1 - u_2)^2 + (v_1 - v_2)^2]\, dx}\,, \quad (3)$$

as a function of time. As can be seen, the synchronization error tends to zero as soon as the coupling is switched on.

Figure 2 shows an X-T diagram: a region below the curve is the region of synchronicity. In other words, for each X (T) there exists a critical value T_{cr} (X_{cr}) such that for all $T < T_{\mathrm{cr}}$ ($X < X_{\mathrm{cr}}$) drive and response systems are synchronized. In the case when $T > \Delta t$ both systems (1) and (2) are coupled at discrete times only. For example, for $X = (8/256)L$, $T_{\mathrm{cr}} = 80\Delta t$; therefore, for all $\Delta t < T < T_{\mathrm{cr}}$ the synchronization is achieved with *discretely sampled signals.*

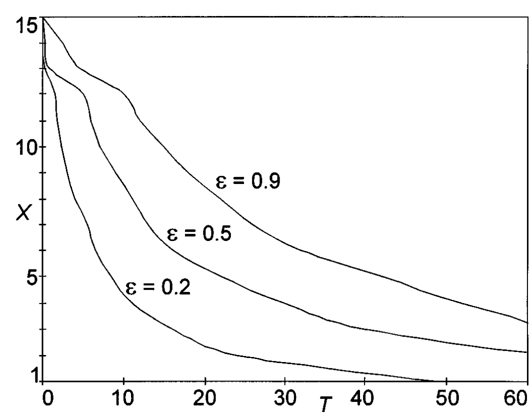

FIG. 2. X-T diagram for the Gray-Scott model.

We point out that the two PDEs can also be synchronized in the following way: at time $t = kT$ only a single space point $x = mX$ is influenced by setting the v_2 variable to a new value $v_2(mX, kT) = v_2(mX, kT - 0) + \varepsilon[v_1(mX, kT) - v_2(mX, kT - 0)]$. Then, at time $t = (k + 1)T$, the space point $x = (m + 1)X$ is driven, etc. Therefore, the driving signal is the sequence of samples $v_1(0,0), v_1(X, T), v_1(2X, 2T), \ldots, v_1(0, NT), v_1(X, (N + 1)T), v_1(2X, (N + 2)T), \ldots$. For example, if $X = (8/256)L$, $T = 5\Delta t$, and $\varepsilon = 0.9$, synchronization of two infinite dimensional systems can be achieved by a *single scalar signal.*

The drive and the response systems synchronize, because the spectrum of Lyapunov exponents [15] of the solution generated by the response system has values which are less than 0. Therefore, the synchronization manifold $u_1(x,t) = v_1(x,t)$ and $u_2(x,t) = v_2(x,t)$ is linearly stable. This is exactly the same condition for synchronization as for low-dimensional systems given in terms of conditional Lyapunov exponents [2]. For example, for the parameters from the synchronization region (Fig. 2) all Lyapunov exponents of the response system are negative. The fact that the response and drive systems are synchronized with a finite number of points can be used for a simple and intuitive measure of degree of spatiotemporal chaos in PDEs: if synchronization can be achieved with a smaller number of space points, then spatiotemporal chaos is weaker. We will elaborate this idea in detail elsewhere.

How general is this method for synchronization of spatiotemporal chaos? We have performed numerical experiments with different PDEs (complex Ginzburg-Landau equation [15,16], one-dimensional nonlinear drift-wave equation driven by a sinusoidal wave [17], Kuramoto-Sivashinsky equation [18]), and below we present as a second example the one dimensional complex Ginzburg-Landau equation (CGLE) in the regime of spatiotemporal amplitude chaos [15,16],

$$\frac{\partial a_1}{\partial t} = a_1 - (1 - i\beta)\,|a_1|^2 a_1 + (1 + i\alpha)\,\frac{\partial^2 a_1}{\partial x^2}\,. \quad (4)$$

We use this equation to drive another CGLE,

$$\frac{\partial a_2}{\partial t} = a_2 - (1 - i\beta)\,|a_2|^2 a_2 + (1 + i\alpha)\,\frac{\partial^2 a_2}{\partial x^2} + f(x,t)\,. \quad (5)$$

For the numerical integration periodic boundary conditions are imposed, i.e., $a_i(x,t) = a_i(x + 40\pi, t)$. We use an explicit Euler scheme with 1024 points, and time steps equal to 0.0002. The values of the parameters are $\alpha = 4$, $\beta = 4$, $\varepsilon = 0.8$, $X = (16/1024)40\pi$, and $T = 30\Delta t$. The results are similar to the case of the Gray-Scott model; namely, the synchronization error approaches zero as soon as the coupling is switched on (see Fig. 3). Similar results have also been obtained with other PDEs. We note here that very recently Sushchik [19] has investigated synchronization of spatiotemporal chaos in a pair of CGLEs. However, in his work only the case when all space points are continuously driven was considered, that is, the case when $X = T = 0$.

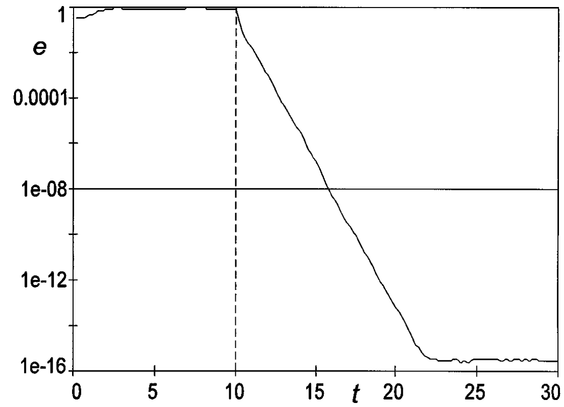

FIG. 3. The synchronization error Eq. (3) versus time for the complex Ginzburg-Landau equations (4) and (5).

We now discuss a possible application of the synchronization method introduced in this Letter. It is known that synchronization of unidirectionally coupled systems and control of chaos are equivalent concepts [20]. Therefore, it is not surprising that one can use the above method for

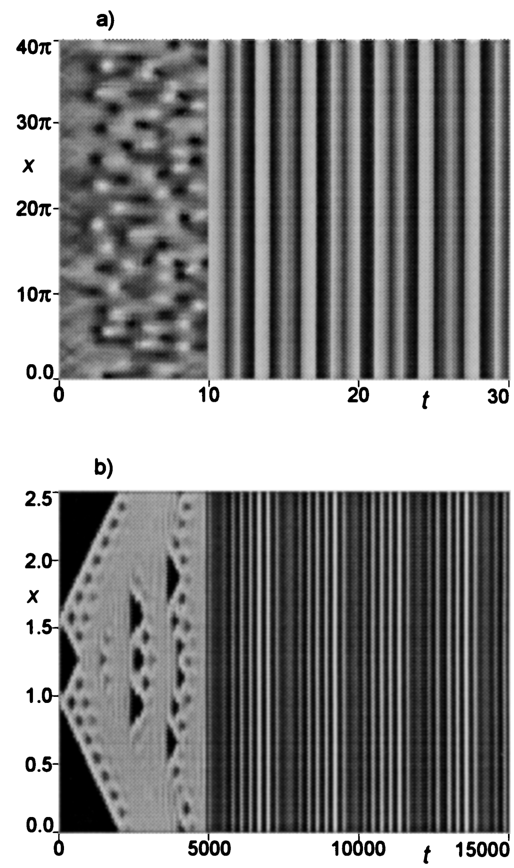

FIG. 4. (a) Control of the complex Ginzburg-Landau equation; $X = (10/1024)40\pi$, $T = 20\Delta t$, $\varepsilon = 0.8$. (b) Suppression of chaos in the Gray-Scott model; $X = (4/256)L$, $T = 10\Delta t$, $\varepsilon = 0.2$.

control [21] and suppression [22] of chaos as well. For example, if the driving signal a_1 in Eq. (5) is replaced by $\exp(i\beta t)$ the spatiotemporal solution of (5) quickly collapses to oscillations that are homogeneous in space and periodic in time, as it is shown in Fig. 4(a). Similarly, the function $v_1 = 0.2 \sin(0.02t)$ can be used to suppress chaos and establish a regular pattern in the Gray-Scott model [Fig. 4(b)].

To conclude, the synchronization method proposed in this Letter can be applied to a pair of unidirectional coupled PDEs of different types. The synchronization is achieved by applying the driving signals only at a finite number of space points. Our approach is very general and can be useful for practical applications whenever one needs to synchronize spatiotemporal systems. Synchronization of chaos is currently suggested for communication applications. We expect that the possibility of coding information in both time and spatial chaotic states will have much wider and deeper application prospects than in temporal systems [3,4].

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