Measuring the State of a Bosonic Two-Mode Quantum Field

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A measurement scheme is proposed to determine the state of a bosonic two-mode system completely. Without relying on the existence of a reference field, we reconstruct the quantum state from joint number measurement only, in case $[\boldsymbol{\rho}, \hat{N}_{\text{tot}}] = 0$. Based on an analogy to angular momentum, we have obtained an explicit inversion procedure for the density matrix of the system and discuss its application. [S0031-9007(97)04719-4]

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Tomographic methods already had a well established history in many areas of applied classical physics, e.g., medical imaging, geological sciences, or signal processing, before it was recognized that the quantum mechanical state is also an "object" that can be viewed from different directions. Recording the information that is gained from a particular measurement from all possible view points of the object determines a quantum state uniquely.

For the measurement of the state of a magnetic dipole (a discrete degree of freedom) Newton and Young [1] introduced the concept of rotating quantum states into tomographic physics. More recently, it was applied in the context of cavity QED [2], as well. However, state rotation is also the center piece of *phase space tomography* [3,4], which is applicable to systems with a continuous degree of freedom.

The latter method has found many beautiful experimental realizations such as the pioneering measurement of the state of a single quantized cavity mode [5,6]. Successively, it has been applied to the study of molecular states [7], motional states of trapped ions [8–12], and atomic beams [13]. The method has even found its way back to classical diffractive optics [14] as it is based on the superposition principle and the uncertainty relation.

By extending the original ideas of state measurement to systems with more than 1 degree of freedom [15,16], or to systems that consist of identical particles, tomography provides a systematic tool to study the implications of state entanglement or aspects of quantum statistics. This is of particular importance as the controlled generation of entangled states, e.g., Schrödinger cat states or maximally entangled GHZ states [17,18], as well as Bose-Einstein condensation (BEC) of dilute alkali gases [19,20] are both current research topics of central interest and experimentally feasible.

Thus, tomography can be used in a complementary manner to the proposed schemes of engineering quantum states [21] in the laboratory and to determine quantum correlations of noninteracting massive condensates, unambiguously.

The purpose of the present Letter is to present a novel state measurement scheme for an *a priori* unknown density matrix of a bosonic two-mode field,

$$\boldsymbol{\rho} = \sum_{n_0,n_1,n_0',n_1'=0}^{\infty} |n_0, n_1\rangle\langle n_0', n_1'| \rho_{n_1 n_1'}^{n_0 n_0'} . \tag{1}$$

In here $\{|n_0, n_1\rangle\}$ represents a two-mode number state basis with $\hat{N}_{\text{tot}}|n_0, n_1\rangle = (n_0 + n_1)|n_0, n_1\rangle$.

Different measurement procedures have to be devised whether (i) $[\boldsymbol{\rho}, \hat{N}_{\text{tot}}] \neq 0$ or (ii) $[\boldsymbol{\rho}, \hat{N}_{\text{tot}}] = 0$. The general case (i) is characterized by the existence of coherence between total number states. A measurement that probes such coherences necessarily connects unequal number manifolds. For optical photons this is easily accomplished by a heterodyne measurement, i.e., by mixing the examined modes with an external coherent reference laser field. Various such schemes have been studied in the past and applied successfully in the optical domain. However, the use of analogous methods in quantum statistical mechanics is precluded as there are no coherent reference fields known at present. Moreover, situation (ii), i.e., states that have a sharp total particle number or incoherent superpositions of such states, occurs more naturally for atomic ensembles. Thus, we will focus on this case (ii) and determine the quantum state of an ensemble of bosonic atoms from joint count rates only. It will suffice to determine the probability distribution

$$P_{\mathcal{R}}(n_0, n_1) = \langle n_0, n_1|U_{\mathcal{R}}\boldsymbol{\rho}U_{\mathcal{R}}^{\dagger}|n_0, n_1\rangle, \tag{2}$$

of obtaining n_0 and n_1 counts in modes a_0 and a_1, respectively. This has to be repeated for a finite sequence of different measurements on identically prepared ensembles. The set of required unitary operations $U_{\mathcal{R}}$ will be specified later on in this Letter [22].

This is of relevance for two mode condensates [19,20], e.g., two trapped magnetic sublevels in an external potential ground state or two external modes of motion with equal internal states. In the absence of two particle interactions (adiabatic state expansion) such a measurement scheme can determine all quantum correlations of the system from particle number measurements only.

One possible realization of such a scheme is shown in Fig. 1. Two orthogonal position modes $|x\rangle$, $|y\rangle$ are selected by a double slit, and the field $\hat{\psi}_\zeta = \delta(\zeta - x)\hat{\psi}_x + \delta(\zeta - y)\hat{\psi}_y$ propagates downstream through two separate atomic beam splitters, e.g., adiabatic momentum transfer,

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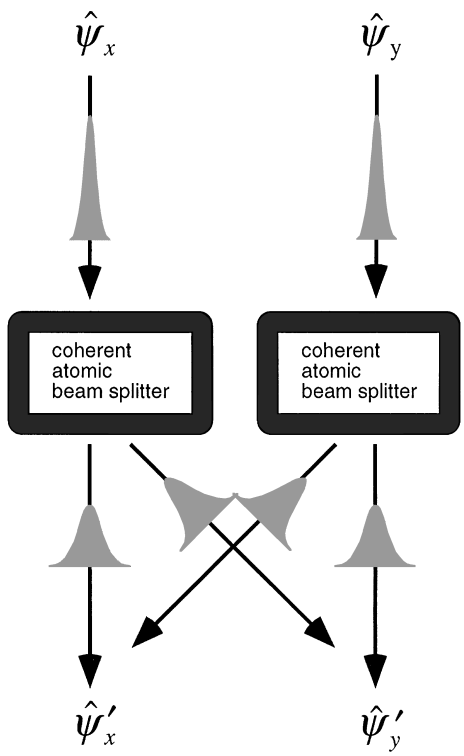

FIG. 1. Two orthogonal position modes $|x\rangle$, $|y\rangle$ of a massive field $\hat{\psi}$ are selected by a double slit. As the particles propagate downstream, they are coherently superimposed by atomic beam splitters and collected by two detectors.

magnetic mirrors, or Bragg scattering. At the output side each of the modes will have a coherent admixture of the other, i.e., $\hat{\hat{\psi}}_x = \cos\beta\hat{\psi}_x + e^{-i\gamma}\sin\beta\hat{\psi}_y$. By adjusting the relative phase γ of the partial fractions and, finally, by observing the occupation number of the modes, it is possible to determine the quantum state.

The analogous setup for two orthogonal photonic modes is depicted in Fig. 2. As one of the modes propagates through a phase shifter, it accumulates an adjustable phase difference with respect to the other. Successively, both modes are mixed by a nonbalanced beam splitter, and joint photon count rates of the output modes are recorded.

The required linear transformations $U_{\mathcal{R}}$ are given by polarization rotators or beam splitters in the case of the photon field or, on the other hand, by any single particle interactions, that mix two modes of a massive condensate. A generalization of this procedure to an arbitrary number of modes seems feasible by applying the required unitary transformations successively to all possible pairs [23].

The measurement procedure is essentially based on the Jordan-Schwinger analogy of harmonic oscillators with the quantum description of angular momentum [24]. For this purpose, we will consider four bosonic modes $\{a_0, a_1, b_0, b_1\}$ with $[a_i, a_j^\dagger] = \delta_{ij}$, $[b_i, b_j^\dagger] = \delta_{ij}$ that are related by a unitary transformation

$$\begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = U_{\mathcal{R}}^\dagger \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} U_{\mathcal{R}} . \quad (3)$$

This transformation could describe the time evolution of a

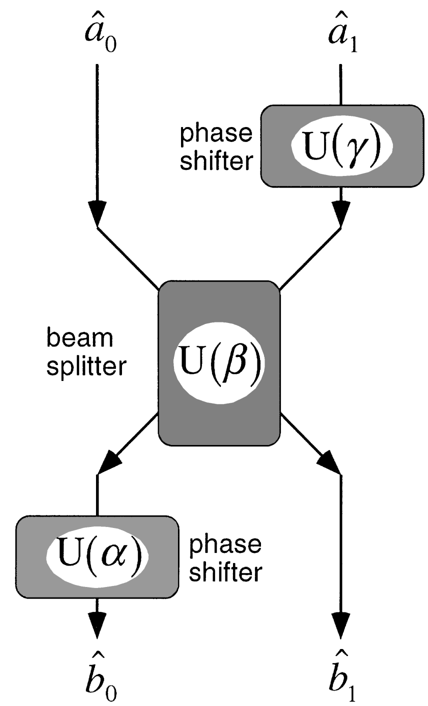

FIG. 2. Setup of an optical beam splitter. Two optical modes a_0 and a_1 of orthogonal propagation direction or polarization are phase shifted $[U(\gamma), U(\alpha)]$ and mixed coherently in a general nonbalanced beam-splitter device $[U(\beta)]$.

dynamical system $[a_i = a_i(t_{\text{initial}}),\, b_i = a_i(t_{\text{final}})]$ or represent an input-output relationship between different mode sets ($a_i = a_i^{\text{in}}$, $b_i = a_i^{\text{out}}$). The constraint to preserve the commutation relations also fixes the number of excitations within the system, i.e., $[U_{\mathcal{R}}, \hat{N}_{\text{tot}}] = 0$. Following Schwinger's concept of angular momentum, one can define the rotation group by a set of bilinear combinations of creation and annihilation operators

$$\hat{L}_1 = \frac{1}{2}(a_0^\dagger a_1 + a_1^\dagger a_0), \qquad \hat{L}_2 = \frac{1}{2i}(a_0^\dagger a_1 - a_1^\dagger a_0), \quad (4)$$

$$\hat{L}_3 = \frac{1}{2}(a_0^\dagger a_0 - a_1^\dagger a_1), \qquad [\hat{L}_i, \hat{L}_j] = i\epsilon_{ij}^k \hat{L}_k . \quad (5)$$

The analogy with angular momentum becomes complete if a square operator $\hat{\mathbf{L}}^2$ is defined according to

$$\hat{\mathbf{L}}^2 = \sum_{i=1}^{3} \hat{L}_i^2 = \hat{l}(\hat{l} + 1), \qquad \hat{l} = \frac{1}{2}(\hat{n}_0 + \hat{n}_1), \quad (6)$$

and a certain quantization axis $\mathbf{e}_3$ is chosen to select a projection operator $\hat{m} = \mathbf{e}_3 \cdot \hat{\mathbf{L}} = \frac{1}{2}(\hat{n}_0 - \hat{n}_1)$. The eigenvectors $|l, m\rangle_{\mathbf{e}_3}$ of this complete set of commuting observables (CSCO) $\{\hat{\mathbf{L}}^2, \hat{m}\}$ are

$$|l, m\rangle_{\mathbf{e}_3} \equiv |n_0 = l + m\rangle_{a_0} \otimes |n_1 = l - m\rangle_{a_1} . \quad (7)$$

By choosing a rotated quantization axis, e.g., $\mathbf{v} = \mathrm{R}\mathbf{e}_3$, one obtains a different set of basis states $|l, m\rangle_{\mathbf{v}}$ that are labeled by the CSCO: $\{\hat{\mathbf{L}}^2, \mathbf{v} \cdot \hat{\mathbf{L}}\}$. They induce an l-dimensional representation of the rotation group

$$|l,m\rangle_{\mathcal{R}\mathbf{e}_3} = U_{\mathcal{R}}|l,m\rangle_{\mathbf{e}_3} = \sum_{m'=-l}^{l} D^{(l)}_{m'm}(\mathcal{R})\,|l,m'\rangle_{\mathbf{e}_3}\,. \quad (8)$$

To be specific, we will adopt the Euler parametrization [25] for the unitary rotation operators $U_{\mathcal{R}(\alpha,\beta,\gamma)}$ and the corresponding Wigner matrices $D^{(l)}_{m'm}(\mathcal{R}(\alpha,\beta,\gamma))$

$$U_{(\alpha,\beta,\gamma)} = e^{-i\alpha\hat{L}_3}e^{-i\beta\hat{L}_2}e^{-i\gamma\hat{L}_3}, \quad (9)$$

$$D^{(l)}_{m'm}(\alpha,\beta,\gamma) = {}_{\mathbf{e}_3}\langle l,m'|U_{(\alpha,\beta,\gamma)}|l,m\rangle_{\mathbf{e}_3} = e^{-im'\alpha}d^{(l)}_{m'm}(\beta)e^{-im\gamma}. \quad (10)$$

The key idea of the measurement scheme is based on a representation of the initial density matrix Eq. (1) with respect to the angular momentum states

$$\boldsymbol{\rho} = \sum_{l,l'=0}^{\infty}\sum_{m=-l}^{l}\sum_{m'=-l'}^{l'} |l,m\rangle_{\mathbf{e}_3}\langle l',m'|_{\mathbf{e}_3}\rho^{ll'}_{mm'}\,. \quad (11)$$

A temporal evolution or a unitary transformation between different modes in a beam splitter maps an initial (input) state $\boldsymbol{\rho}$ onto a final (output) state $\overline{\boldsymbol{\rho}}$,

$$\overline{\boldsymbol{\rho}}(\alpha,\beta,\gamma) = U_{\mathcal{R}(\alpha,\beta,\gamma)}\boldsymbol{\rho}U^{\dagger}_{\mathcal{R}(\alpha,\beta,\gamma)}\,. \quad (12)$$

If the diagonal elements of Eq. (12) are taken with respect to the angular momentum basis

$$P^L_M(\beta,\gamma) = {}_{\mathbf{e}_3}\langle L,M|\overline{\boldsymbol{\rho}}(\alpha,\beta,\gamma)\,|L,M\rangle_{\mathbf{e}_3}\,, \quad (13)$$

one obtains an equation reminiscent of a *discrete Fourier transform*

$$P^L_M(\beta,\gamma) = \sum_{\nu=-2L}^{2L} e^{i\gamma\nu}X^{L\nu}_M(\beta)\,, \quad (14)$$

$$X^{L\nu}_M(\beta) = \sum_{m=-L}^{L} d^{(L)}_{Mm}(\beta)d^{(L)}_{M,\nu+m}(\beta)\rho^{LL}_{m,\nu+m}\,. \quad (15)$$

A final phase shift α is not observable from a number measurement, thus vanishes identically.

In the following, we are going to show that Eq. (14) can be inverted with respect to the initial density matrix $\boldsymbol{\rho}$, if all of the probabilities $\{L,M,k\|P^L_M(\beta,\gamma_k)\geq 0\}$, can be determined for $|k|\leq 2L$ different rotation angles $\gamma_k = 2\pi k/(4L+1)$.

The inversion of Eq. (14) is based on two steps. First, by introducing a discrete Fourier transform of the probability $P^L_M(\beta,\gamma_k)$, we get

$$X^{L\mu}_M(\beta) = \frac{1}{4L+1}\sum_{k=-2L}^{2L} e^{-i\gamma_k\mu}P^L_M(\beta,\gamma_k)\,,$$

for $|\mu|\leq 2L$. So far, the inclination angle β is an arbitrary constant.

Before inverting Eq. (15), it is worthwhile to note an identity that is derived from the addition theorem for Wigner matrices as well as orthogonality relations for Clebsch-Gordan coefficients [1,25]

$$\delta_{m\overline{m}} = \sum_{j=0}^{2L}\sum_{M=-L}^{L} C^{LLj}_{-\overline{m},\mu+\overline{m},\mu}C^{L,L,j}_{-M,M,0} \times \frac{(-1)^{m-M}}{d^{(j)}_{0,\mu}(\beta)}\,d^{(L)}_{Mm}(\beta)d^{(L)}_{M,\mu+m}(\beta)\,. \quad (16)$$

Finally, if this is applied to Eq. (15), one finds

$$\rho^{LL}_{m,\mu+m} = \sum_{j=0}^{2L}\sum_{M=-L}^{L}\frac{(-1)^{m-M}}{d^{(j)}_{0,\mu}(\beta)}\,C^{LLj}_{-m,\mu+m,\mu}C^{LLj}_{-MM0} \times \frac{1}{4L+1}\sum_{k=-2L}^{2L} e^{-i\gamma_k\mu}P^L_M(\beta,\gamma_k)\,. \quad (17)$$

In short, this relation links probabilities $P^L_M(\beta,\gamma_k)$ that are determined experimentally from joint count rates $P_{\mathcal{R}(0,\beta,\gamma_k)}(n_0,n_1)$ [Eq. (2)] to the density matrix of an *a priori* unknown state $\boldsymbol{\rho}$. The inclination angle β that was arbitrary so far should not coincide with the zeros of associated Legendre polynomials that are proportional to $d^{(j)}_{0,\mu}(\beta)\neq 0$.

To illustrate this method, we have assumed that the initial state $\boldsymbol{\rho}$ of the two-mode system is in an incoherent superposition of a pure (Schrödinger cat) state $|\psi(N)\rangle = (|N,0\rangle + |0,N\rangle)/\sqrt{2}$ and a thermal background $\boldsymbol{\rho}_{\rm th}(\langle\hat{N}_{\rm tot}\rangle)$, characterized by a mean excitation $\langle\hat{N}_{\rm tot}\rangle$ and a thermodynamic potential $\Omega_{N_{\rm tot}}$. Furthermore, we assumed for simplicity that the two modes are energy degenerate, i.e., $\hat{H} = \hbar\omega\hat{N}_{\rm tot}$

$$\boldsymbol{\rho}(N,\langle\hat{N}\rangle) = \frac{1}{2}\left(|\psi(N)\rangle\langle\psi(N)| + e^{\Omega_{N_{\rm tot}}-(\hat{N}_{\rm tot}/\langle\hat{N}_{\rm tot}\rangle)}\right). \quad (18)$$

Such a state, i.e., $\rho(N=7,\langle\hat{N}_{\rm tot}\rangle=3)$ is represented in Fig. 3. It is important to note that this particular state has no off-diagonal elements with respect to a total particle number, i.e., $[\boldsymbol{\rho},\hat{N}_{\rm tot}]=0$. Therefore, it will be sufficient to observe joint count rates of the output channels. The embedding Hilbert space is restricted to a total particle number of $N_{\max}=8$ and is formed by $\dim(\mathcal{H}_{N_{\max}})=45$ basis states. The basis states were arranged in a lexicographic

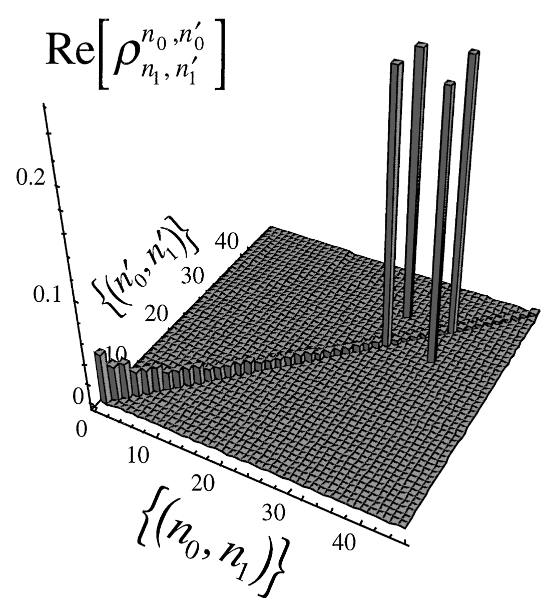

FIG. 3. Initial density matrix $\boldsymbol{\rho}^{n_0n_0'}_{n_1n_1'}$ of a bosonic two-mode system vs the ordered pairs of quantum numbers $\{(n_0,n_1)\}$ and $\{(n_0',n_1')\}$.

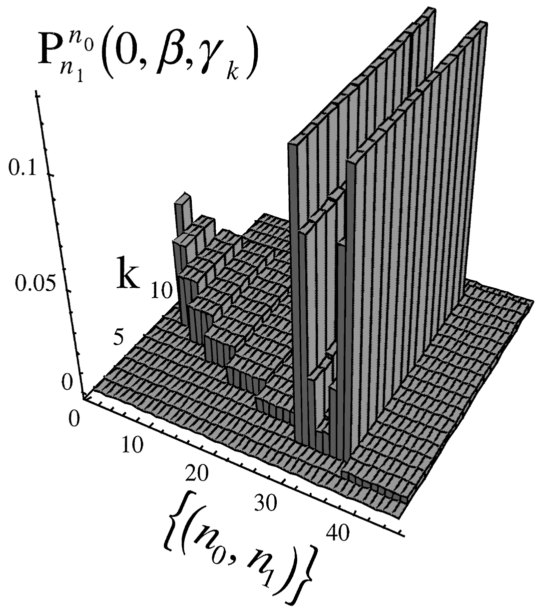

FIG. 4. The joint probability distribution $P_{\mathcal{R}(0,\beta,\gamma_k)}(n_0,n_1)$ for measuring n_0 counts in mode b_0 and n_1 counts in mode b_1 plotted vs the index of the phase angles γ_k and the ordered pairs of quantum numbers. The phase angles were equidistant, and the inclination angle was chosen arbitrarily as $\beta = \pi/5$.

order along the coordinate axes, i.e., $(n_0,n_1) > (n_0',n_1')$ if $n_0 > n_0'$ or $n_0 = n_0'$ and $n_1 > n_1'$.

We have simulated the outcome of an experiment numerically by evaluating joint count probabilities $P_{\mathcal{R}(0,\beta,\gamma_k)}(n_0,n_1)$ [Eq. (13)] for various phase angles γ_k from the given initial state $\rho(N = 7, \langle \hat{N}_{\text{tot}} \rangle = 3)$. This is shown in Fig. 4. With this set of data and by applying the inversion theorem Eq. (17), the initial density matrix can be recovered completely.

We have also tested the reconstruction procedure with respect to an initial density matrix that was chosen at random $\boldsymbol{\rho}_{\text{rand}}(N_{\max} = 8)$. The only constraints were $\text{Tr}[\boldsymbol{\rho}_{\text{rand}}] = 1, \boldsymbol{\rho} \geq 0$ (positive semidefinite) and $[\boldsymbol{\rho}_{\text{rand}}, \hat{N}_{\text{tot}}] = 0$. Again, we obtained a faithful image of the original state.

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Note added.—After completing this work, we became aware of a similar proposal [26]. The authors reconstructed the density matrix for a constant particle number numerically and notice a singular behavior for certain values of θ.

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[1] R.G. Newton and B. Young, Ann. Phys. (N.Y.) **49**, 393 (1968).
[2] R. Walser, J.I. Cirac, and P. Zoller, Phys. Rev. Lett. **77**, 2658 (1996).
[3] J. Bertrand and P. Bertrand, Found. Phys. **17**, 397 (1987).
[4] K. Vogel and H. Risken, Phys. Rev. A **40**, R2847 (1989).
[5] D.T. Smithey, M. Beck, M.G. Raymer, and A. Faridani, Phys. Rev. Lett. **70**, 1244 (1993); M.G. Raymer, J. Cooper, and M. Beck, Phys. Rev. A **48**, 4617 (1993).
[6] P.J. Bardroff, E. Mayr, and W.P. Schleich, Phys. Rev. A **51**, 4963 (1995); P.J. Bardroff, E. Mayr, W.P. Schleich, P. Domokos, M. Brune, J.M. Raimond, and S. Haroche, Phys. Rev. A **53**, 2736 (1996).
[7] T.J. Dunn, I.A. Walmsley, and S. Mukamel, Phys. Rev. Lett. **74**, 884 (1995); I.A. Walmsley and M.G. Raymer, Phys. Rev. A **52**, 681 (1995).
[8] S. Wallentowitz and W. Vogel, Phys. Rev. Lett. **75**, 2932 (1995).
[9] U. Leonhardt, Phys. Rev. Lett. **74**, 4101 (1995).
[10] R. Blatt, J.I. Cirac, and P. Zoller, Phys. Rev. A **52**, 518 (1995).
[11] J.F. Poyatos, R. Walser, J.I. Cirac, P. Zoller, and R. Blatt, Phys. Rev. A **53**, R1966 (1996).
[12] D. Leibfried, D.M. Meekhof, B.E. King, C. Monroe, W.M. Itano, and D.J. Wineland, Phys. Rev. Lett. **77**, 4281 (1996).
[13] U. Janicke and M. Wilkens, J. Mod. Opt. **42**, 2183 (1995).
[14] A.W. Lohmann, J. Opt. Soc. Am. A **10**, 2181 (1993); M.G. Raymer, M. Beck, and D.F. McAlister, Phys. Rev. Lett. **72**, 1137 (1994).
[15] H. Kühn, D.G. Welsch, and W. Vogel, Phys. Rev. A **51**, 4240 (1995).
[16] M.G. Raymer, D.F. McAlister, and U. Leonhardt, Phys. Rev. A **54**, 2397 (1996).
[17] L. Davidovich, M. Brune, J.M. Raimond, and S. Haroche, Phys. Rev. A **53**, 1295 (1996).
[18] A. Zeilinger, M.A. Horne, H. Weinfurter, and M. Zukowski, Phys. Rev. Lett. **78**, 3031 (1997).
[19] C.J. Myatt, E.A. Burt, R.W. Ghrist, E.A. Cornell, and C.E. Wieman, Phys. Rev. Lett. **78**, 586 (1997).
[20] M.-O. Mewes, M.R. Andrews, D.M. Kurn, D.S. Durfee, C.G. Townsend, and W. Ketterle, Phys. Rev. Lett. **78**, 582 (1997).
[21] C.K. Law and J.H. Eberly, Phys. Rev. Lett. **76**, 1055 (1996); J.F. Poyatos, J.I. Cirac, and P. Zoller, Phys. Rev. Lett. **77**, 4728 (1996).
[22] A generalization of the presented method to include case (i) is possible if a single mode quadrature and a number measurement on the other channel can be performed as well, i.e., $P_{\mathcal{R}}(n_0,n_1,n_1') = \langle n_0,n_1|U_{\mathcal{R}}\boldsymbol{\rho}U_{\mathcal{R}}^{\dagger}|n_0,n_1'\rangle$.
[23] M. Reck, A. Zeilinger, H. Bernstein, and P. Bertani, Phys. Rev. Lett. **73**, 58 (1994).
[24] B. Yurke, S.L. McCall, and J.R. Klauder, Phys. Rev. A **33**, 4033 (1986); R.A. Campos, B.E. Saleh, and M.C. Teich, Phys. Rev. A **40**, 1371 (1989).
[25] L.C. Biedenharn and J.D. Louck, *Angular Momentum in Quantum Physics*, Encyclopedia of Mathematics and Its Applications Vol. 8 (Addison-Wesley Publishing Company, Reading, MA, 1981).
[26] E.L. Bolda, S.M. Tan, and D.F. Walls, previous Letter, Phys. Rev. Lett. **79**, 4719 (1997).