Attractor-Repeller Collision and Eyelet Intermittency at the Transition to Phase Synchronization

Arkady Pikovsky, Grigory Osipov, Michael Rosenblum, Michael Zaks, and Jürgen Kurths
Department of Physics, University of Potsdam, Am Neuen Palais, PF 601553, D-14415, Potsdam, Germany
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The chaotically driven circle map is considered as the simplest model of phase synchronization of a chaotic continuous-time oscillator by external periodic force. The phase dynamics is analyzed via phase-locking regions of the periodic cycles embedded in the strange attractor. It is shown that full synchronization, where all the periodic cycles are phase locked, disappears via the attractor-repeller collision. Beyond the transition an intermittent regime with exponentially rare phase slips, resulting from the trajectory's hits on an eyelet, is observed. [S0031-9007(97)03524-2]

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Studies of the transitions from regular to chaotic behavior have demonstrated might of concepts of scaling, renormalization, and universality in nonlinear dynamics [1]. Transitions within chaos require the incorporation of statistical methods. Good examples are the crisis [2,3] resulting in a sudden change of the attractor size, and the chaos-hyperchaos transition [4]. Another nontrivial transition in chaos is the symmetry-breaking bifurcation [5], which describes in particular the onset of complete synchronization in interacting chaotic systems [6,7]. Statistical properties of the modulational intermittency appearing at the symmetry-breaking transition have been studied in [5,8].

Recently, phase synchronization of chaotic oscillators has been investigated theoretically [9–11] and experimentally [9,12,13]. In a periodically forced chaotic system the phase synchronization appears as the frequency entrainment by the external force: The mean frequency of chaotic oscillations (calculated, e.g., as a number of maxima of the chaotic process per unit time) is locked by the external frequency. This synchronization corresponds to the appearance of the "phase order" while the amplitude remains irregular.

In this paper we demonstrate that the onset of phase synchronization corresponds to a special transition in chaotic systems: a collision of an attractor with a repeller. Near the collision a specific intermittency is observed, appearing as extremely rare phase slips resulting from leakages through an "eyelet" [14].

As a model for investigating the attractor-repeller collision we study the following two-dimensional mapping

$$x(t+1) = f(x(t), \phi), \tag{1a}$$

$$\phi(t+1) = \phi(t) + \Omega + \varepsilon \cos[2\pi\phi(t)] + g(x(t)). \tag{1b}$$

From the mathematical viewpoint this is a system of the circle map coupled to the chaotic map f. For concreteness, we use here as a representative example the perturbed tent map

$$f(x, \phi) = 1 - a|x| + \varepsilon\rho \sin[2\pi\phi(t)].$$

Physically, the system (1) models the dynamics of a continuous-time chaotic oscillator under periodic external force. The mapping (1) should be interpreted as a stroboscopic mapping taken at each period of the force. The variable x describes the amplitude of the oscillator, and ϕ corresponds to its phase (in our normalization this phase varies between 0 to 1). The parameter ε is related to the amplitude of the force; in the autonomous case $\varepsilon = 0$ the dynamics of the phase is according to (1b) purely diffusive with zero Lyapunov exponent and does not influence the chaotic dynamics of the amplitude, as can be expected for an autonomous continuous-time chaotic oscillator [15]. The detuning between the period of oscillations and the period of the external force is described by two terms: $g(x(t))$ corresponds to nonuniformity of phase rotations in the autonomous chaotic oscillator, while the term Ω is proportional to the frequency of the external force. Albeit the small force influences both the phase and the amplitude, its effect on the phase is physically more important because of locking phenomena, while the effect on the chaotic amplitude is small because a chaotic attractor is relatively stable to perturbations.

We emphasize that our consideration of the forced chaotic oscillator is fully analogous to the usual description of synchronization and phase locking in periodic oscillators via the circle map [16,17] [in the latter case the term $g(x(t))$ is a constant]. The main difference to the periodic case is in the term $g(x)$ which describes a chaotic modulation of the phase motion. In some approaches this term was approximated by Gaussian random noise, which allows one to describe the phase synchronization qualitatively [9]. Here we consider this term dynamically and show that it leads to an attractor-repeller collision and a special eyelet intermittency. For simplicity, in this Letter we take the driving term in the simple form $g(x) = \delta x$. Also, below we consider only the case of small forcing $\varepsilon < (2\pi)^{-1}$ so that without chaotic forcing no phase chaotization can happen.

To characterize the synchronization, in full analogy with the purely periodic case, we define the phase rotation number ω as an average growth rate of the phase

$$\omega = \lim_{N\to\infty} \frac{\langle \phi(N) - \phi(0) \rangle}{N} .$$

This quantity can be interpreted as a difference between the mean frequency of the forced chaotic oscillator and that of the external force. The dependence of the rotation number on the external frequency Ω (Fig. 1) characterizes the phase synchronization: the region where $\omega = 0$ corresponds to the entrainment of the chaotic oscillator frequency by the external force frequency.

Our approach to analyze this kind of synchronization is based on the representation of a chaotic attractor through unstable periodic trajectories embedded in it (see [17] for general properties of unstable periodic orbits inside chaotic attractors and [5,18] for discussions of metamorphoses of these unstable orbits at other types of synchronization). First, let us characterize the periodic orbits in the autonomous case ($\varepsilon = 0$). Each periodic trajectory can be characterized by its real period T, and by its "symbolic" integer period N which counts, roughly speaking, the number of rotations the trajectory performs (or, more precisely, the number of iterations of the corresponding Poincaré map). In general $T \approx T_0 N$ where T_0 is the average return time; the deviations are, however, very important. In our stroboscopic representation (1) the term $g(x)$ in (1b) describes these deviations, so that while the tent map (1a) has periodic orbits of all "symbolic" periods, generally there are no periodic orbits in the full system (1), because the phase rotations are generally incommensurate with the period of external force.

We now apply the external force and follow the unstable periodic trajectories. For each such trajectory, in full analogy to the synchronization of stable periodic oscillations [17,19], a phase-locked region appears. In the terms of the system (1) this means that for each periodic orbit in (1a) we can construct the main phase-locked region with rotation number $\omega = 0$. Some of these regions, which are nothing else but Arnol'd tongues, are shown in Fig. 2. The tongues stick into different points on the $\varepsilon = 0$ line, because different periodic orbits of the chaotic oscillator have different periods. For each orbit with "symbolic" period N $x(1), \dots, x(N)$ of the tent map (1a) the dynamics of the phase variable ϕ inside the phase-locked region is simple: There exist a corresponding stable $\phi_s(1), \dots, \phi_s(N)$ and an unstable $\phi_u(1), \dots, \phi_u(N)$ orbit (this is stability in the ϕ direction, all these orbits are of course unstable in the x direction). At the border of synchronization these orbits disappear via the saddle-node bifurcation and a state with nonzero rotation number appears.

A region where all the phase-locked regions overlap is the grey one in Fig. 2. It is bounded by the phase-locked regions of the periodic orbits having the maximal and the minimal average period T/N, for the set of parameters of Fig. 2 these are the fixed point and a period-4 cycle. In this region all periodic orbits embedded in the chaotic attractor are locked, with corresponding stable and unstable orbits of (1) shown in Fig. 3(a). These orbits can be considered as skeletons of the attractor and the repeller, respectively, and they are well separated. All trajectories on the attractor wander in a vicinity of the skeleton, therefore the value of the phase remains bounded, and the rotation number is exactly zero. Indeed, it is zero for each periodic orbit embedded in the attractor, and therefore zero for each trajectory on the attractor as the latter can be approximated with a periodic one. We call this domain the region of full phase synchronization.

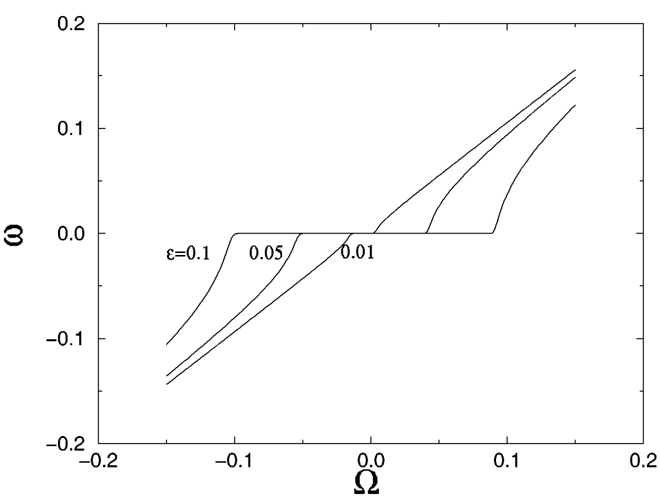

FIG. 1. Dependence of the rotation number ω on the external parameter Ω for the model (1) with $\delta = 0.05$, $\rho = 0.05$, $a = 1.9$, and different values of forcing: $\varepsilon = 0.1$, $\varepsilon = 0.05$, and $\varepsilon = 0.01$.

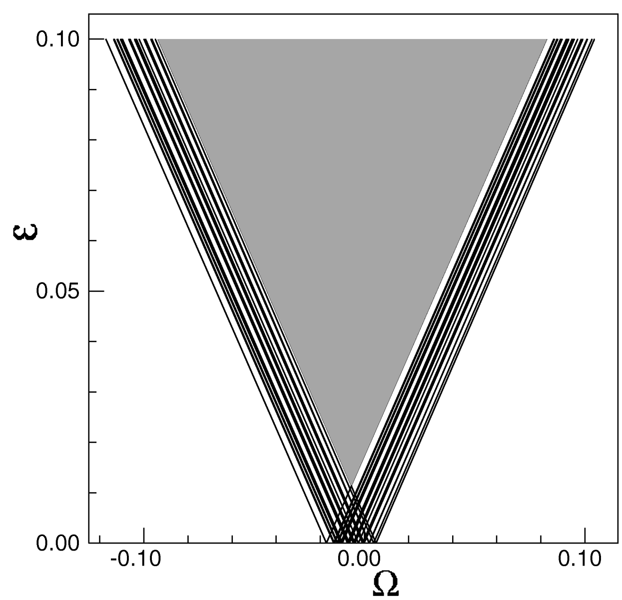

FIG. 2. Phase-locking regions for the periodic orbits with periods 1–5 for the same parameters as in Fig. 1. The region of full phase synchronization, where all the phase-locking regions overlap, is delineated with grey.

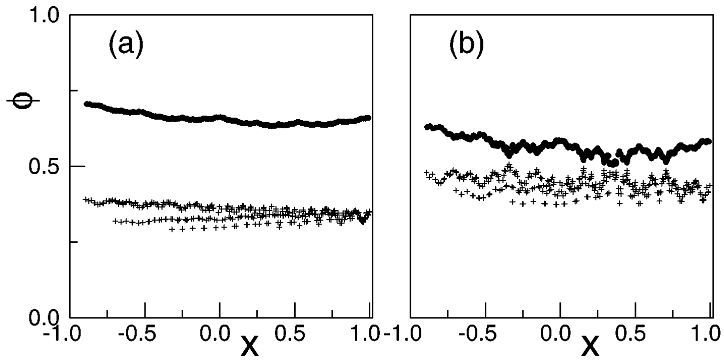

FIG. 3. The stable (pluses) and unstable (filled circles) periodic orbits with periods 1–8 forming the skeletons of the attractor and repeller, respectively. (a) Inside the full synchronization region $\varepsilon = 0.1$, $\delta = 0.05$, $\Omega = 0.05$, $\rho = 0.05$, $a = 1.9$, the attractor and the repeller are distinct. (b) Just after the attractor-repeller collision, at which the stable and unstable (in ϕ direction) fixed points disappear, $\Omega = 0.085$.

As the parameters of the system are changed in such a way that the boundary of the region of full phase synchronization is approached, the attractor and the repeller come close to each other. At the transition point of attractor-repeller collision the saddle-node bifurcation of one of the unstable periodic orbits occurs. The situation just beyond the transition is shown in Fig. 3(b). Although most cycles remain phase locked, those few, which have lost phase locking, allow phase slips (at a slip the phase changes by ± 1) to occur. We now develop a statistical theory of these slips (cf. [14]).

Consider, for definiteness, the transition at which the fixed point belonging to the attractor collides with the fixed point belonging to the repeller. At the bifurcation point, one multiplicator (approximately in the direction of ϕ, we call it weakly unstable direction) is one and the other (approximately in the direction of x, we call it strongly unstable direction) is larger than one in the absolute value, we denote it μ. Then the dynamics on the weakly unstable direction is the same as for usual saddle-node bifurcation, with a characteristic time of phase slip (at which the phase changes by ± 1) growing as an inverse square root of the distance to the bifurcation point, in the same way as at the type-I intermittency [20]:

$$\tau_{sl} \approx C_1(\Omega - \Omega_c)^{-1/2}. \tag{2}$$

For a chaotic trajectory such a phase slip can also occur, if the trajectory of the map (1) stays for a long time—at least τ_{sl}—in a close vicinity of the weakly unstable direction. Because the other direction is strongly unstable, the distance to the weakly unstable direction $\Delta(t)$ grows exponentially in time $\Delta(t) \sim \Delta(0)\,|\mu|^t$, so this distance should be initially very small to allow at least one slip to occur:

$$\Delta(0) < C_2|\mu|^{-\tau_{sl}}. \tag{3}$$

This region is exponentially small, like an "eyelet," and the phase slips are correspondingly extremely rare [14]. Using the nearly uniform invariant probability density for the tent map we can estimate the probability to visit any interval as proportional to its length. Thus, the probability for a phase slip to occur is proportional to the right-hand side of (3), and the rotation number is proportional to this probability. As a result, we obtain the following expression for the rotation number at the attractor-repeller collision transition [14]:

$$\log(\omega) \sim -(\Omega - \Omega_c)^{-1/2}. \tag{4}$$

We check this relation in Fig. 4. It is valid for both transitions where the system leaves the phase-locked region. From the consideration above it is clear that the time statistics of phase slips corresponds to the statistics of Poincaré recurrence times for a chaotic system [statistics of the returns to the eyelet (3)], and this is known to have the exponential tail [21]. From the relation (4) it is also clear what the main difference is of the eyelet intermittency to the other types, e.g., to the intermittency

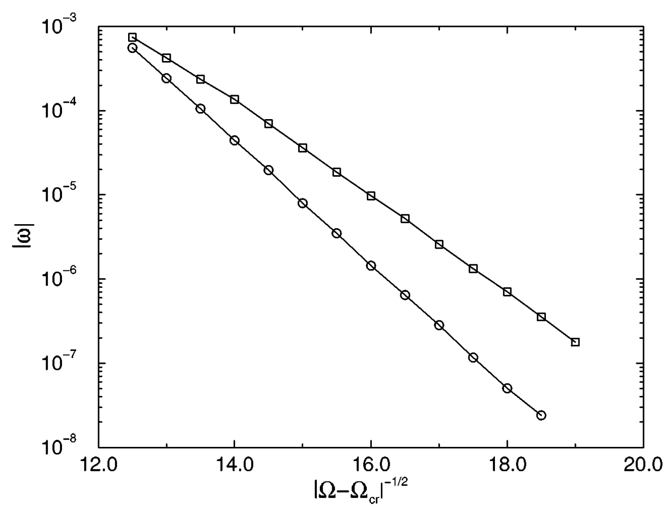

FIG. 4. The rotation number ω at the borders of the attractor-repeller transition in the region of eyelet intermittency. The parameters are $\delta = 0.05$, $\varepsilon = 0.1$, $\rho = 0.05$, $a = 1.9$.

at the crisis: Here the intermittent bursts are exponentially rare near the transition point, while for other types their probability grows as a power law [2,20].

The exponentially slow eyelet intermittency is the reason why the phase-locked region for the chaotically driven circle map (Fig. 1) appears to be larger than the region of full phase synchronization, and why nearly perfect phase synchronization can be observed also for small amplitudes ε, for which there is no full phase synchronization at all. Only when a sufficiently large number of periodic orbits undergoes a saddle-node bifurcation and the probability of phase slip becomes large, does one observe a deviation of the mean observed frequency from the frequency of the external force.

We have considered the simplest possible case when the borders of the region of full phase synchronization are given by the phase-locking regions of the fixed or periodic points. A more complex situation can occur if an extremum is reached on a chaotic everywhere dense trajectory. Then the attractor and the repeller can collide in a dense set of points; a similar situation happens in a quasiperiodically forced circle map [22]. This latter case needs special investigation.

In conclusion, we have considered phase synchronization transition using as a model the chaotically forced circle map. The region of full phase synchronization is defined as the overlap of phase-locking tongues for all periodic cycles embedded in chaos; in this region an attractor and a repeller exist corresponding to stable and unstable values of the phase. At the boundary of this region the attractor in the circle map collides with the repeller, and phase slips become possible. For a slip to occur, a trajectory has to pass through an extremely tiny eyelet appearing at the collision points, thus the slips are exponentially rare. In this Letter we restricted our consideration to the simplest discrete-time model (1). We have also calculated the phase locking of unstable periodic orbits embedded in the continuous-time Rössler attractor; these results qualitatively agree with the consideration above and are reported elsewhere [23].

The transition described can be put in a general framework of bifurcations of strange attractors. Here the unstable direction is not affected, but in the transversal stable direction the attractor undergoes a "saddle-node" bifurcation. Similar to other bifurcations of strange attractors (e.g., the symmetry-breaking bifurcation discussed in [5,24] can be described as a "pitchfork" one) this transition is not abrupt but smeared. The best way to see this is to follow different unstable periodic orbits embedded in a strange attractor: Each of these orbits undergoes the standard saddle-node bifurcation, but at different values of parameters. The transition starts when the first orbit bifurcates and ends when this happens with the last orbit; the whole infinite set of particular simple bifurcations composes the nontrivial transition of the strange attractor.

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