Model for Subharmonic Waves in Granular Materials

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Parametric waves arise spontaneously when a thin layer of granular material is oscillated vertically. A dynamical model of these waves accounting for the observed instability threshold and appearance of square patterns is presented. The proposed instability mechanism is due to two competing processes: a focusing effect that concentrates particles in space, and a diffusion effect that relaxes large thickness gradients in the layer. When the layer is in a liquid-type state, close to a solid-liquid transition, the model correctly predicts the presence of oscillons recently reported by P. Umbanhowar *et al.* [Nature (London) **382**, 793 (1996)]. [S0031-9007(97)04636-X]

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Recent experiments examining thin layers of vertically vibrated granular material have revealed that parametric surface waves occur when the dimensionless acceleration $\Gamma = A(2\pi f)^2/g$ exceeds a critical value [1,2] (here g is the acceleration of gravity, A is the amplitude of the vibrating surface, and f is the frequency of the driving force). The primary instability gives rise to squares or stripes depending on the frequency f and particle diameter d. It was found that the frequency at the center of the range in which the square to stripe transition occurs is proportional to $d^{-1/2}$. This scaling is understood in terms of the ratio of kinetic energy injected into the layer to the potential energy required to raise a particle a fraction of its diameter, that is, v_0^2/gd (where v_0 is the vertical velocity relative to the neighboring particle). Considering that Γ is constant and v_0 is proportional to the container velocity (so that $v_0 \sim f^{-1}$), then at low f, $v_0^2/gd \gg 1$, the horizontal mobility is high since the layer dilation δ is large and the dissipation is small; while at large f, $v_0^2/gd \ll 1$, the layer dilation is small and the dissipation is large. Only stripes are observed at high f when the energy injection is low. In this regime, δ is too small to compare to the particle diameter to allow relative motion of the grains. The pattern wavelength decreases with frequency and saturates at a constant value at a very high frequency [1,3,4].

Another important fact is that, in the same domain of frequencies where the squares to stripes transition occurs, solitary waves or "oscillons" are observed for Γ smaller than the instability threshold of waves [5]. For oscillons to exist, the primary bifurcation must be hysteretic so that the waves and the flat layer can coexist at the same parameter values. In the experiment, hysteresis is large for low f and small for high f. Oscillons, however, appear only in a range of f where the hysteresis decreases with increasing frequency. It was suggested that the strong increase in dissipation occurring when δ becomes smaller than a fraction of particle diameter is responsible for keeping structures localized [5].

Here, we present a model explaining the instability onset of granular surface waves. Our model is based on the fact that lateral motion of grains at large scale takes place when the layer is in free flight, whereas, thickness relaxation occurs only during the layer-plate collision. A master equation is introduced to take into account the mass motion while simple arguments allow us to write down a diffusion equation for the layer thickness. Numerical simulation shows that square patterns are selected by the nonlinearities introduced in this model and localized structures are observed when the local dilation in the layer becomes small with respect to the particle diameter.

Figure 1 is a side view of the experimental cell showing the time evolution of the waves close to the layer-container collison. During the collision, thickness modulations relax until the layer takes off. Each layer modulation splits into two sand packets which move laterally. They subsequently collide and form new modulations shifted in space by half the pattern wavelength. The pattern is then subharmonic, oscillating at one half the drive frequency. Lateral motion of grains takes place during every cycle of the external excitation. This motion is similar to that of a fluid particle in a standing surface wave. However, there is one important difference: In the fluid case, the lateral transfer of mass is induced by the gradient of hydrostatic pressure which exists at all times [6]. In the granular case, the pressure gradient exists only during the collision and is a result of the transfer of momentum from the plate to the grain network. In our analysis, we, therefore, consider the effective drive to be composed of a series of impulsive accelerations that only occur when the layer collides with the plate.

The dynamics of the layer described in the continuum limit using two variables: $h(\vec{x}, t)$, the local layer thickness, and $\vec{v}_\perp(\vec{x}, t)$, the horizontal component of the grain velocity [$\vec{x}$ represents the horizontal coordinates (x, y) and t is the time]. We consider $\vec{v}_\perp(\vec{x}, t)$ to be independent of the vertical coordinate. With these variables, the

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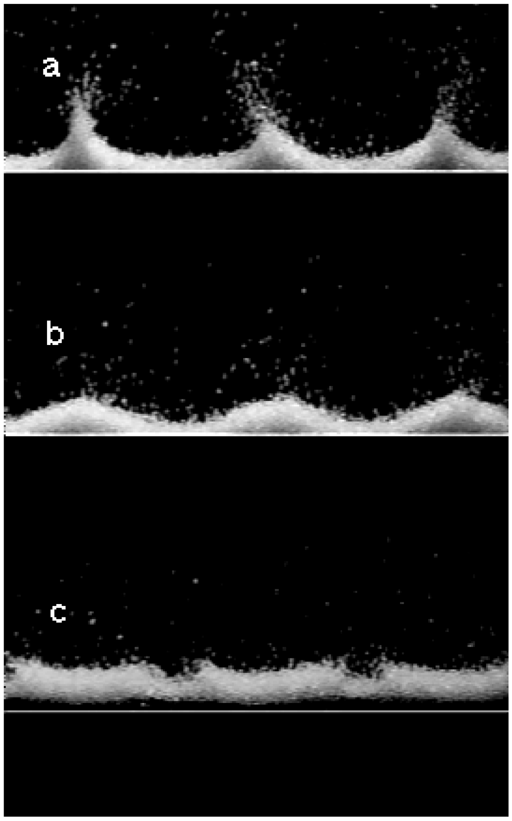

FIG. 1. Side view of a parametric wave obtained in a narrow cell. The horizontal white line indicates the moving plate. (a) and (b) show the wavy pattern immediately before the collision. (c) After the collision, lateral transfer of grains occurs.

continuity equation that accounts for mass conservation reads

$$h(\vec{x}, t_f) = h(\vec{x}, 0) + \int d\vec{r}[W(\vec{r} \to \vec{x})h(\vec{r}, 0) - W(\vec{x} \to \vec{r})h(\vec{x}, 0)], \quad (1)$$

where $W(\vec{r} \to \vec{x})$ is the probability density for the motion of a single column of sand from an initial position $\vec{r}$ to a final position $\vec{x}$. For simplicity, we consider deterministic trajectories by taking $W(\vec{r} \to \vec{x})$ as the Dirac-δ function, that is

$$W(\vec{r} \to \vec{x}) = C\delta\{\vec{x} - [\vec{r} + t_f \vec{v}_\perp(\vec{r}, 0)]\}, \quad (2)$$

which reflects the fact that a fraction C ($0 < C \leq 1$) of the grains that were at position $\vec{r}$ at $t = 0$ move to a new position $\vec{v}_\perp(\vec{x}, 0)t_f$, where $\vec{v}_\perp(\vec{x}, 0)$ is the lateral grain speed induced during the collision and t_f is the characteristic time during which the lateral motion of grains takes place. Therefore, t_f is the flight time and also the characteristic time for the kinetic energy injection into the layer. Equation (2) is an approximation; in reality there is a distribution of grain speeds and here we consider only the mean lateral velocity of such a distribution. Our approximation is valid when the characteristic horizontal length, the wavelength of the pattern λ, is appreciably larger than the thickness of the layer $h(\vec{x}, t)$. As a consequence of momentum conservation, the mean lateral speed, right after the collision, is proportional to both the layer-plate collision velocity v_0 and the slope of the free surface of the layer [7]

$$\vec{v}_\perp(\vec{x}, 0) = -v_0 \nabla_\perp h(\vec{x}, 0), \quad (3)$$

where $\nabla_\perp = (\partial_x, \partial_y)$.

Our next step is to introduce a mechanism to relax the high thickness gradient in the layer. Such gradients are a result of the accumulation of sand in regions where fluxes of sand meet. A simple way to implement this idea is through a diffusion equation which automatically conserves mass [8,9]:

$$\partial_t h(\vec{x}, t) = D\nabla_\perp^2 h(\vec{x}, t), \quad (4)$$

where t ranges in the interval $t_f < t < T$, T is the time between successive collisions with the plate, and D is a diffusion coefficient that reflects dissipative processes which cause the layer slope to spread out. In our case, diffusion of thickness is induced when particles move down the slope at the free surface of the layer. An estimate of the dimensional dependence of D can be obtained by considering the flowing layer of grains to be a fluid with kinematic viscosity ν. In our case, the velocity of the grains should be taken as proportional to v_0 and the thickness gradient. The total flux of particles $\vec{j}$ flowing in the layer is then

$$\vec{j} \sim v_0 f(N, e, \Gamma)(d + \delta)\nabla_\perp h, \quad (5)$$

where $f(N, e, \Gamma)(d + \delta)$ is a viscous length which takes into account the penetration of the grain flow into the layer and $f(N, e, \Gamma)$ is an unknown function of $N = h/d$, restitution coefficient e and Γ. Introducing this expression into the mass conservation equation for the grains, $\partial_t h + \nabla_\perp \cdot \vec{j} = 0$, we identify the diffusion coefficient as $D \sim v_0 f(N, e, \Gamma)(d + \delta)$. At this point, we notice that D is proportional to the kinematic viscosity $\nu \sim (d + \delta)^2/t_c \sim v_0(d + \delta)$ of the layer [10]. The main reason for this result rests in the fact that the only characteristic speed in our problem is the collision speed v_0, the lack of an intrinsic thermal speed being a direct consequence of the strong energy dissipation occurring in the granular layer.

We now study the stability of a vibrated layer by performing a linear stability analysis. We define $h(\vec{x}, t) = h_0 + \xi(\vec{x}, t)$, where h_0 is the thickness of the flat layer and $\xi(\vec{x}, t)$ is a disturbance in the height. Using (1), (2), (3), and (4), and writing $\xi(\vec{x}, t) = \Phi_k(t)e^{i\vec{k}\cdot\vec{x}}$ + c.c. in terms of a perturbation wave vector $\vec{k}$ and an amplitude $\Phi_k(t)$, we obtain

$$\Phi_k(T) = \sigma(k)\Phi_k(0), \quad (6)$$

with

$$\sigma(k) = (1 - Ch_0 v_0 t_f k^2)e^{-D(T - t_f)k^2}, \quad (7)$$

which means that after each collision, the amplitude of mode k, Φ_k is amplified by $\sigma(k)$ and grows to $\sigma(k)^n$ after

n collisions. Thus, if $|\sigma(k)| > 1$, the system is unstable; using Eq. (7), the only way to obtain amplification of disturbances is for $\sigma(k) < -1$. This result says that at each collision the phase of the unstable mode must change by a factor π, changing the sign of the disturbance. Such a phase change is characteristic of a subharmonic standing wave, since after two collisions the perturbation returns to its original phase. The instability onset and the most unstable wave vector are then determined for the conditions $\sigma'(k_*) = 0$ and $\sigma(k_*) = -1$. This occurs when the natural control parameter of the instability $\tilde{\Gamma} = Ch_0 v_0 t_f / D(T - t_f)$ exceeds a critical value $\tilde{\Gamma}_c \approx 3.6$ for $k_* = 1.2/\sqrt{D(T - t_f)}$.

Using the natural choice for C as the ratio of the penetration length to the layer thickness, $C = f(N, e, \Gamma)(d + \delta)/h_0$, we obtain $\tilde{\Gamma} \approx t_f/(T - t_f)$, which fits nicely with the control parameter that was shown to be pertinent in experiments [2]. $\tilde{\Gamma}$ has an interesting physical meaning, since it shows a balance between energy injected into and energy dissipated in the layer. Thus, $\tilde{\Gamma}$ is similar to the parameter controlling the onset for Faraday's instability in a highly viscous fluid [11]. The critical value of $\tilde{\Gamma}$ predicted here, however, differs from experimental measurements, as should be expected when numerical coefficients have been disregarded (for instance, when the dependence of $\tilde{\Gamma}$ on the restitution coefficient and friction has not been considered explicitly).

On the other hand, the most unstable wavelength at the instability onset λ varies like $[f(N, e, \Gamma)(d + \delta) v_0 t_f / \tilde{\Gamma}]^{1/2}$. This result tells us that two regimes are possible for λ depending on whether δ is much smaller or much larger than d. Thus, a crossover occurs naturally when δ becomes of the order of d and corresponds to the change of the behavior of λ with f observed in experiments close to the square to stripe transition. In addition, our model predicts that λ selected is a viscous length, this is, $\lambda \sim (\nu t_f/\tilde{\Gamma})^{1/2}$. It should be noticed that Faraday's instability in viscous fluids also selects a wavelength of viscous type $\lambda \sim (\nu/f)^{1/2}$ [12]. However, in the fluid case, ν is not a function of f.

In order to check if the model gives the observed scaling for λ, we need an estimate of the functional dependence of δ with f. Unfortunately, we have neither an accurate experimental method nor a procedure to estimate δ from first principles. Nevertheless, numerical simulation in a ball column [13] shows that δ should be a function of the kinetic energy injected into the layer and the number of collisions necessary to dissipate it, i.e., $\delta \sim v_0^2/N^{5/2} g$, where $N = h_0/d$. This estimate implies that, in the low frequency regime ($\delta \gg d$), λ scales like $1/f^2$, in agreement with experimental results obtained in glass particles [4]. However, other scaling for λ is possible depending on the restitution coefficient of particles and friction [4].

The nonlinear regime of waves can be studied by solving Eqs. (1), (2), (3), and (4) numerically in two spatial dimensions. The procedure is the following: First, as in Eq. (1), the grain column of height $f(N, e, \Gamma)(d + \delta)$ and base $d\vec{r}$ is translated and added at a point $\vec{r} - v_0 t_f \nabla_\perp h(\vec{r}, t)$ and second, we apply the diffusion process Eq. (4) until the next taking off point, i.e., during a time $(T - t_f)$. For low values of $\tilde{\Gamma}$, the homogeneous state $h(\vec{x}, t) = h_0$ is stable. However, for $\tilde{\Gamma} = \tilde{\Gamma}_c$, parametric waves appear as predicted by the linear stability analysis. Squares are observed in the numerical simulation [see Fig. 2(a) and, in contrast with experimental observations, the squares transition is always supercritical.

One possible way to account for subcriticality is by introducing in our model the internal friction angle α of the granular layer. One can estimate α by putting the flying distance ($v_0 t_f |\nabla h| \approx v_0 t_f \alpha$) of the order of the particle size, thus, $\alpha \approx d/v_0 t_f$ which gives the good behavior of α for large $v_0 t_f$ [14]. However, as $v_0 t_f$ becomes of the same order of d, the contact between grains becomes more important and the friction cannot be neglected, thus, α goes to angle of repose α_c for $v_0 t_f \ll d$. This dependence could be represented by a tentative formula: $\tan(\pi\alpha/2\alpha_c) \sim d/v_0 t_f$. In our case, α acts as a minimum angle of activation for which the lateral motion of the grains is allowed. Thus, in the numerical procedure, a grains column is moved laterally only when the free surface slope exceeds α.

Numerical results show, in this case, that squares are subcritical with hysteresis increasing with α. Figure 2(b) illustrates a well developed squared pattern and squares coexisting with a flat layer state in the subcritical region of squares. Because of subcriticality, one hopes that localized structures or pulses appear and, due to the nonvariational features of this system, these solitary waves will be stable structures. These correspond to the oscillons reported recently in Ref. [5]. For C near the unity, the pulses are only a small transient disappearing very quickly; however, as C diminishes, for $\tilde{\Gamma}$ remaining constant and f increasing as in experiments, the lifetime of pulses and molecules is much longer. (In some sense, the parameter C makes the "liquid-solid" transition: $C \approx 1$

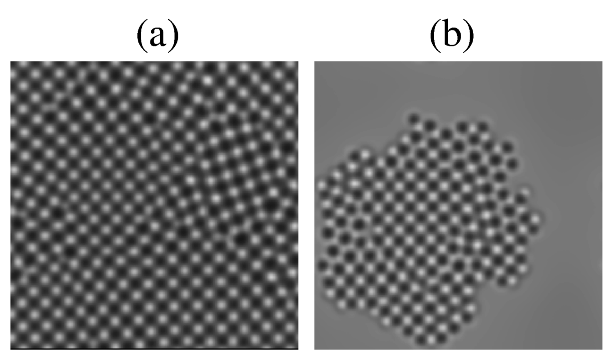

FIG. 2. (a) Square pattern obtained by numerical simulation realized for $\tilde{\Gamma} = 5$ and $C = 1$. (b) A metastable regime dominated by the competition of a square pattern and the flat layer, obtained by numerical simulation for $\tilde{\Gamma} = 5$, $C = 1$, and $\tan\alpha \approx 0.1$. The spatial resolution of both simulations is $dx = 0.5[D(T - t_f)]^{1/2}$.

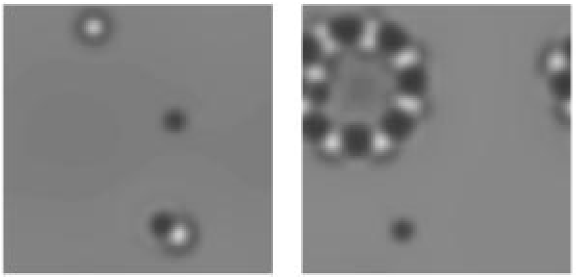

FIG. 3. Subharmonic isolated solitary waves for $C = 0.77$, $\tilde{\Gamma} = 8$, and $\tan\alpha \approx 0.1$.

for a liquid, while the solid arises for $C \approx 0$.) Figure 3 shows a collection of different situations, from elementary pulses to more complex configurations or molecules. According to Ref. [5], the subharmonic pulses are at rest, having two phases characterized by a hill and a crater. The dynamics and behavior agree quite well with the experiments.

In conclusion, we find that an intuitive model accounts for several features of parametric waves and gives valuable insight into the instability mechanism. When a disturbance appears in the free surface of the layer, a lateral flux is induced due to the layer-plate collision. Consequently, depending on the shape of the disturbance, regions where grains accumulate can exist. In fact, if we focus on particles located at position $\vec{r}$ (the disturbance region) and go to a position $\vec{x} = \vec{r} - v_0 t_f \vec{\nabla} h(\vec{r})$ (flat region) after a time t_f, and we impose mass conservation $h(\vec{x}) d^2x \sim h(\vec{r}) d^2r$, we find that a self-focusing effect occurs when $d^2x < d^2r$. That happens for a downhill of grains ($\nabla^2 h > 0$) because the volume at the new position is $d^2x \approx d^2r(1 - v_0 t_f \nabla^2 h)$. The disturbance will thus be larger at $\vec{x}$ than it was at $\vec{r}$. This self-focusing mechanism always leads to instability with no spatial scale selection. However, processes of thickness diffusion allow us to select a wavelength of the wave pattern. In this sense, our model differs from the one presented in Ref. [15], in which wave pattern is a result of a dynamical nonuniformity of the forcing oscillation. In our case, the competition between self-focusing and thickness diffusion determines that the parameter controlling the instability is essentially the acceleration of the plate, in agreement with experimental results. In addition, the squared pattern result of a subcritical instability is correctly described by introducing the internal friction angle of a granular material. For instance, the wavelength dependence on Γ observed experimentally can be related to the variation of α with changes in Γ [14]. Finally, the model presented here also reproduces the existence and the rules of combination of solitary waves reported by Umbanhowar *et al.* [5].

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