Towards a Field Theory of Fractional Quantum Hall States

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We present a Chern-Simons theory of the fractional quantum Hall effect in which flux attachment is followed by a transformation that effectively attaches the correlation holes. We extract the correlated wave functions, compute the drift and cyclotron currents (due to inhomogeneous density), exhibit the Read operator and operators that create quasiparticles and quasiholes. We show how the bare kinetic energy can get quenched and replaced by one due to interactions. We find that for $\nu = 1/2$ the low energy theory has neutral quasiparticles and give the effective Hamiltonian and constraints. [S0031-9007(97)04676-0]

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The experimental discovery of the fractional quantum Hall effect [1,2] led to theoretical response on two fronts: trial wave functions that captured the essential physics and approximate computational schemes starting with the microscopic Hamiltonian. The most successful among the latter has been the Chern-Simons (CS) field theory [3–16]. Here we present a formulation of the CS theory which resolves several nagging questions and exposes the physics in a particularly transparent way. We illustrate our method through the cases $\nu = 1/3$ and $\nu = 1/2$, where the filling fraction $\nu = 2\pi n/eB$, n being the particle density, $-e$ the electron charge, and B the magnetic field down the z axis. Our results for $\nu = p/(2np + 1)$ will be reported later. We set $\hbar = c =$ volume $= 1, z = x + iy$, and $l_0 = (eB)^{-1/2}$ the cyclotron length.

In the CS approach one introduces a wave function for the CS particles in terms of the electronic one as follows:

$$\Psi_e = \prod_{i<j} \frac{(z_i - z_j)^l}{|z_i - z_j|^l} \Psi_{\rm CS} . \tag{1}$$

where l is the number of flux quanta to be attached. The prefactor introduces a gauge field $\mathbf{a}$ obeying

$$\frac{\nabla \times \mathbf{a}(\mathbf{r})}{2\pi l} = \sum_i \delta(\mathbf{r} - \mathbf{r}_i) . \tag{2}$$

In second quantized form, the CS action density is

$$S = \overline{\psi} i \partial_o \psi + a_0 \left(\frac{\nabla \times \mathbf{a}}{2\pi l} - \overline{\psi}\psi \right) - \frac{|(-i\nabla + e\mathbf{A} + \mathbf{a})\psi|^2}{2m} , \tag{3}$$

where $\mathbf{A}$ is the external vector potential and m is the bare mass. The Coulomb interaction will be added later. By shifting $\mathbf{a}$ we can cancel $e\mathbf{A}$ upon choosing $l = 3$ for $\nu = 1/3$ and $l = 2$ for $\nu = 1/2$. Hereafter $\mathbf{a}$ and $\psi^\dagger \psi = \rho$ (the density) will denote normal-ordered quantities. Whereas in the quest for wave functions, we can build in not just the phase, but all of $\prod (z_i - z_j)^l$, or even the ubiquitous Gaussian factors into the process of flux attachment, doing so here would lead to a complex vector potential [17]. The correlation zeros must therefore be extracted out of the fluctuations [13].

We now introduce our scheme (inspired by the work of Bohm-Pines [18]) and define a composite particle (CP) field, where P may stand for fermion F or boson B:

$$\psi_{\rm CS}(x,y,t) = \exp\left[-i \int_{-\infty}^{t} a_0(x,y,t')\, dt' \right] \psi_{\rm CP}(x,y,t) . \tag{4}$$

The transformation kills the $a_0 \overline{\psi}\psi$ term and introduces a longitudinal vector potential $2\pi l \mathbf{P}$ defined by

$$\mathbf{q} a_0(\mathbf{q},\omega) = -\omega 2\pi l \mathbf{P}(\mathbf{q},\omega) \tag{5}$$

in the kinetic energy term, while $a_0(\frac{\nabla\times\mathbf{a}}{2\pi l})$ becomes

$$\sum_q \int d\omega P(-\omega,-q)[-i\omega] a(q,\omega) , \tag{6}$$

where $P = -i\hat{\mathbf{q}} \cdot \mathbf{P}$ and $a = i\hat{\mathbf{q}} \times \mathbf{a}$, *so that the longitudinal and transverse vector potential are now canonically conjugate and the constraint field has become dynamical.* The Hamiltonian density is

$$H = \frac{1}{2m} |(-i\nabla + \mathbf{a} + 2\pi l \mathbf{P})\psi|^2 \tag{7}$$

$$= \frac{1}{2m}|\nabla\psi|^2 + \frac{n}{2m}(a^2 + 4\pi^2 l^2 P^2) + (\mathbf{a} + 2\pi l \mathbf{P}) \cdot \frac{1}{2m} \psi^\dagger (-i\overleftrightarrow{\nabla})\psi + \frac{1}{2m} :\psi^\dagger\psi: (\mathbf{a} + 2\pi l \mathbf{P})^2$$

$$\equiv H_0 + H_{\rm I} + H_{\rm II} , \tag{8}$$

where $H_{\rm I}$ and $H_{\rm II}$ refer to the last two terms. Note that ψ is to be quantized as a boson (fermion) for $\nu = 1/3$ (1/2). *Though $H_{\rm I}$ and $H_{\rm II}$ denote interactions between the particles and the gauge bosons, we are still discussing*

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free electrons. We are, however, paving the way for the Coulomb interaction.

The constraint now defines physical states:

$$\left(\frac{\nabla \times \mathbf{a}}{2\pi l} - :\psi^\dagger \psi:\right) |\text{physical}\rangle = 0\,, \tag{9}$$

ensuring they are singlets under the local gauge symmetry possessed by H and generated by the operator which annihilates physical states above. This constraint is to be expected since we cannot simply add extra degrees of freedom. From H_0 we see that the pair (a, P) describe oscillators at $\omega_0 = eB/m$. Since we started with n electrons in a plane n_0, the number of independent oscillators obeys $n_0 \leq 2n$. *We find that the value $n_0 = n$, i.e., for $0 < q \leq Q = k_F$ recommends itself for many reasons and choose it.* To pay for these degrees of freedom, the particles will be deprived of n coordinates, which will be seen to put them in the lowest Landau level (LLL). Thus in Eq. (7) only the vector potential $\mathbf{a}(q)$ with $0 < q < Q$ will have a conjugate momentum $\mathbf{P}(\mathbf{q})$. For $q > Q$, the short range part $\delta\mathbf{a}(\mathbf{q})$ will be a function of $\rho(q)$ as in Eq. (3). The contribution of $\delta\mathbf{a}(\mathbf{q})$, $H_{\delta a}$, suppressed in Eq. (7), will be discussed later.

We now analyze H first ignoring all but H_0, starting with the case $\nu = 1/3$. In the ground state, the bosons condense into a constant wave function while the oscillators with Hamiltonian

$$H_{\text{osc}} = \sum_q^Q [A^\dagger(q) A(q)] \omega_0\,, \tag{10}$$

where $A(q) = [a(q) + 6\pi i P(q)]/\sqrt{12\pi}$, yield the ground state wave function:

$$\Psi = \exp\left[-\sum_q \frac{1}{12\pi} a^2(\mathbf{q})\right] \tag{11}$$

$$= \exp\left[-\sum_q 3\pi :\rho(\mathbf{q}): \frac{1}{q^2} :\rho(-\mathbf{q}):\right] \tag{12}$$

$$= \prod_{i<j} |z_i - z_j|^3 \exp\left[-\sum_j |z_j|^2/4l_0^2\right] \tag{13}$$

upon using the constraint to get the wave function in terms of particle coordinates. The steps leading to the last line are explained in Kane *et al.* [19] and Zhang's review [13]. Putting back the phase factors from Eq. (1) gives us Laughlin's $\psi_{1/3}$ with the proviso that since $q < Q$ in Eq. (12), our answer is to be trusted only for $|z_i - z_j| > l_0$: as $z_i \longrightarrow z_j$, we know that there are three zeros in a circle of size l_0, but not that they coincide.

Let us understand how not just the phase, but the cubic correlation zeros of $\psi_{1/3}$, got built in. Writing Eq. (4) in operator form (at the origin) [20] as

$$\psi_{\text{CB}}^\dagger = \exp\left[\sum_{\mathbf{q}} \frac{6\pi i}{q} P(\mathbf{q})\right] \psi_{\text{CS}}^\dagger \tag{14}$$

we see that when we create a composite boson, we not only create a CS boson but also displace $a(q)$ by $-6\pi/q$, which, *upon projection to the physical states* leads to a hole of charge -1 (in electronic units). This agrees with Read's picture [21] of how to add an extra electron to the $\nu = 1/3$ state: we add three units of flux, create a correlation hole of charge -1, and drop in the newcomer. Evidently $\psi_{\text{CB}}^\dagger$ is the Read operator that will have a nonzero expectation value in the ground state. Finally, consider

$$\psi_{qh} = \exp\left[\sum_{\mathbf{q}} \frac{2\pi i}{q} P(\mathbf{q})\right], \tag{15}$$

which clearly creates a Laughlin quasihole of charge $-1/3$. (It produces the extra factor $\prod_i |z_i|$, while the phase comes from the vortex in the boson wave function.) The adjoint operator creates a quasielectron. Our procedure, which gives a concrete realization of many ideas pertaining to composite fermions and bosons was possible only upon going to a larger space, where collective charge motion is represented by a, with a conjugate momentum P that can be used to shift it.

We now turn exclusively to $\nu = 1/2$, which was studied exhaustively by Halperin, Lee, and Read (HLR) [15]. First, a similar analysis to the one above yields the Rezayi-Read [22] or Jain wave function (quadratic zeros times the Gaussian, times a Fermi sea) but without any projection to the LLL. The projection will be achieved shortly.

Let us first perform two simple calculations. Imagine adding a smooth weak scalar potential $V(x, y)$. This couples via a term (upon using the constraint)

$$H_V = \int d^2x V \nabla \times \frac{\mathbf{a}}{4\pi}$$

$$= -\int d^2x \frac{\mathbf{a}}{4\pi} \cdot \hat{\mathbf{z}} \times \nabla V\,. \tag{16}$$

This linear coupling in $\mathbf{a}$ shifts $\mathbf{a}$ to a new minimum and leads to a ground state current

$$\langle\, \mathbf{j} \rangle = -\frac{e}{4\pi} \hat{\mathbf{z}} \times \nabla V, \tag{17}$$

which implies a Hall conductance (recall $h/2\pi = 1$)

$$\sigma_{xy} = \frac{e^2}{4\pi} = \frac{e^2}{2h}\,. \tag{18}$$

Although the shifted oscillators are in their ground states, the original ones are in an admixture with excited states, as is essential to get the right response [17].

Next we confirm that any inhomogeneous density $n(x, y)$ leads to an uncanceled cyclotron current [23]

$$\langle\, \mathbf{j}_{\text{cyclo}} \rangle = \frac{e}{2m} \hat{\mathbf{z}} \times \nabla n\,. \tag{19}$$

To this end imagine that there is a spatially varying field $B(x, y)$ and that a suitable scalar potential has been added on top of it to ensure that we are locally at $\nu = 1/2$. *We are thus not calculating any standard response function; our limited goal is to show that this varying density*

$n(x, y)$ *leads to the expected cyclotron current.* Consider the last term H_{II} in Eq. (8)

$$H_{\mathrm{II}} = \frac{1}{2m}\int d^2x : \psi^\dagger\psi : (\mathbf{a} + 4\pi\mathbf{P})^2$$
$$= \frac{1}{2m}\int d^2x : \psi^\dagger\psi : \langle(\mathbf{a} + 4\pi\mathbf{P})^2\rangle + \text{fluctuations}$$
$$= \frac{1}{2m}\int d^2x\left[\frac{1}{4\pi}\nabla\times\mathbf{a}\right]4\pi n(x,y) + \dots \tag{20}$$

where we have used the fact that the zero-point energy density of the $n_0 = n$ oscillators implies

$$\langle(\mathbf{a} + 4\pi\mathbf{P})^2\rangle = 4\pi n(x,y)\,. \tag{21}$$

If we shift the oscillators to the new minimum mandated by this linear term in $\mathbf{a}$, we find an average current given by Eq. (19).

Although answers that only depended on the oscillators were correctly given above in what we call the middle representation, the large kinetic energy of the particles, of order $1/m$, needs to be quenched. This will now be done by eliminating the coupling between the fermions and oscillators by a canonical transformation that takes us to the final representation. We do this approximately by organizing the calculation in powers of q and keeping just the leading terms at each stage; as well as by setting $\sum_i e^{i(k-q)r_i} = n\delta(k-q)$ and dropping the fluctuating part when the density appears in a product with other operators. The full nature of this approximation is unclear to us, especially when $Q = k_F$ and not particularly small. In any event, the results, which have many nice features, are good only for long distances. We drop H_{II} right away since $:\psi^\dagger\psi:$ is explicitly of order q due to the constraint and eliminate H_{I}. The operators (in first quantization) transform as follows (upon dropping vector signs when obvious and defining $V_\pm = V_x \pm iV_y$):

$$\Omega^{\mathrm{old}} = e^{-iS}\Omega e^{iS}, \tag{22}$$

$$iS = \frac{\sqrt{2\pi}}{m\omega_0}\left[\sum_q\sum_i \hat{q}_+ p_{i-} e^{iqr_i}A(q) - \text{H.c.}\right], \tag{23}$$

$$A^{\mathrm{old}}(q) = A(q) - \frac{\sqrt{2\pi}}{m\omega_0}\sum_i \hat{q}_- p_{i+} e^{-iqr_i}, \tag{24}$$

$$\rho^{\mathrm{old}}(q) = \frac{q}{\sqrt{8\pi}}[A(q) + A^\dagger(-q)]$$
$$- il_0^2\sum_i (q\times p_i)e^{-iqr_i}, \tag{25}$$

$$0 = \sum_i e^{-iqr_i} + \frac{il_0^2}{2}\sum_i (q\times p_i)e^{-iqr_i}(\text{const})\,, \tag{26}$$

$$H = \sum_i \frac{p_i^2}{2m} + \sum_{q=0}^{Q}\omega_0 A^\dagger(q)A(q)$$
$$- \frac{1}{2mn}\sum_{q=0}^{Q}\sum_i\sum_j p_i\cdot p_j e^{-iq(r_i-r_j)} + H_{\delta a}\,. \tag{27}$$

(i) The constraint Eq. (26) does not involve the oscillators. As a result, the fermions and oscillators are truly decoupled in this leading approximation and the former face n constraints. These and the Hamiltonian will commute among themselves in an exact canonical transformation since they did before and will commute to leading order in our approximation. The transformations generated by the constraints represent the gauge transformation of the middle representation, except now the particle Hamiltonian must itself be invariant under it. In particular, if we look at the first operator in the constraint (the new density) we see that it generates a shift in all the momenta in the limit $q \longrightarrow 0$. We see H in Eq. (27), is invariant if we once again invoke $\sum_i e^{i(k-q)r_i} = n\delta(k-q)$. This "drifting Fermi" sea (seen by Haldane in his numerical work) is part of the larger gauge symmetry of the particle Hamiltonian. (ii) If we restrict ourselves to the ground state of the oscillators in Eq. (25) we find $\rho^{\mathrm{old}} = \overline{\rho}$ (the second term) which obeys the commutators of magnetic translations in the limit of small q:

$$[\overline{\rho}(q), \overline{\rho}(q')] = il_0^2(q\times q')\overline{\rho}(q+q')\,, \tag{28}$$

an algebra that was studied in detail by Girvin *et al* [24]. Thus we are able to put the electrons in the LLL within a standard field theory by putting our oscillators in their ground states. *Notice that the fermions are now dipolar with respect to electronic charge, a feature that has been anticipated by many authors* [4]. (iii) The third term in Eq. (27) comes from transforming the oscillator part of H_0. The last, $H_{\delta a}$, stands for the $\delta\mathbf{a}$ and $(\delta\mathbf{a})^2$ terms of the short-range gauge field. (iv) The $i = j$ terms in the third sum renormalize $1/m$ downwards, as we decouple the oscillators. We get $1/m^* = 0$ upon using $\sum_q = n$. If we use a smaller Q, there will be a reduction of $1/m*$ to a fraction of $1/m$ (a step in the right direction), but not a full elimination of m dependence in the low energy sector. The choices $Q > k_F$ lead to a negative effective mass and are not viable. (v) The $i \neq j$ terms summed from 0 to Q can be traded for minus the sum from Q to ∞ since they differ by a delta function $\delta(r_i - r_j)$ that vanishes on fermion (and also hard-core boson) wave functions. Let us combine this term with $H_{\delta a}$. These large q variables couple to the fermion whose $1/m^* \longrightarrow 0$ for the choice $n_0 = n$. Integrating out the fermions in RPA, we find that this sector gives the magnetoplasmon with the right position and residue. There is no other structure since $x = \omega/qv^* \longrightarrow \infty$ since $v^* \longrightarrow 0$. (vi) While $1/m = 0$ and the correct cyclotron pole and residue depend on $Q = k_F$ or $n_0 = n$, the dipolar nature of charge, the constraints and the form of the Hamiltonian in Eq. (27) will be valid even if Q is given a smaller value.

Having made the field theory correctly reproduce the quenched fermions and dispersionless magnetoplasmon of the noninteracting problem, we are ready to turn on interactions. We illustrate the procedure with a Coulomb interaction that is cut off at $q = Q$ so that we can treat it entirely in terms of our oscillators. In the final representation of H this adds a term [recall Eq. (25)]

$$H_{\text{Coul}} = \frac{l_0^4}{2} \sum_{ij} \sum_{q=0}^{Q} \frac{2\pi e^2}{q} (q \times p_i)(q \times p_j) e^{-iq(x_i - x_j)} \tag{29}$$

in addition to a term that renormalizes the oscillator frequency to $\omega_0 + e^2 q/4$ and a feeble derivative coupling between the fermions and the oscillator which makes no difference to this order in e^2 or q. The $i = j$ term in H_{Coul} (which describes the interaction between the electron and the correlation hole when they separate) now leads to

$$\frac{1}{m^*} = \frac{e^2 l_0}{6} \equiv C e^2 l_0 \,. \tag{30}$$

Note that $1/m^*$ is not a small q quantity. Our result is just an estimate; loop diagrams will surely give it finite, momentum dependent corrections. It is encouraging that numerical work [25] (using the full Coulomb interaction) gives a not too different value of $C \simeq 0.2$.

The Hamiltonian and constraint are (dropping $H_{\delta a}$):

$$H = \sum_i \frac{p_i^2}{2m^*} + \frac{l_0^4}{2} \sum_{i,j \neq i} \sum_q^Q \frac{2\pi e^2}{q} \times (q \times p_i)(q \times p_j) e^{-iq(r_i - r_j)}, \tag{31}$$

$$0 = \sum_i e^{-iqr_i} + \frac{i l_0^2}{2} \sum_i (q \times p_i) e^{-iqr_i}. \tag{32}$$

The constraint states to this order in q that the density formed out of a putative cyclotron coordinate vanishes, having been spoken for by the oscillators.

One must solve the above theory in a way that respects the constraints, i.e., in a conserving approximation. This has not been done yet to our satisfaction. Given the dipolar nature of charge (the explicit factor of q in ρ^{old}), one may expect that the $\rho^{\text{old}} - \rho^{\text{old}}$ structure factor (and its moments) will have two extra powers of q relative to a Fermi liquid. However, the "drifting sea" might lead to soft modes and compensating inverse powers of q in the low-frequency response function, so that the static compressiblity remain finite in the limit $q \longrightarrow 0$ for short range interactions or vanishes as q for Coulomb interactions.

For $\nu = 1/3$ a similar analysis of the mass renormalization will hold, but the RPA will proceed very differently because the constraint boson mixes with the condensate even at tree level and suppresses low energy excitations.

We have presented a CS theory in which the composite particles carry flux and the correlation holes thanks to the additional transformation that made the CS field into dynamical oscillators, which were then frozen. Depriving the particles of n degrees of freedom led to LLL behavior. We derived the correlated wave functions, drift and cyclotron currents, explicit operators for creating the quasihole and quasiparticle, Read's operator, and properly traded the bare mass for an effective mass based on interactions. For $\nu = 1/2$ we exhibited the dipolar couplings between the final quasiparticles, and derived an effective Hamiltonian and constraints.

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