Breit-Wigner Width and Inverse Participation Ratio in Finite Interacting Fermi Systems

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For many-body Fermi systems we determine the dependence of the Breit-Wigner width Γ and inverse participation ratio ξ on interaction strength $U \geq U_c$ and energy excitation $\delta E \geq \delta E_{\text{ch}}$ when a crossover from Poisson to Wigner-Dyson $P(s)$ statistics takes place. At $U \geq U_c$ the eigenstates are composed of a large number of noninteracting states and even for $U < U_c$ there is a regime where $P(s)$ is close to the Poisson distribution but $\xi \gg 1$. [S0031-9007(97)04709-1]

In 1955 Wigner [1] introduced the local density of states to study "the properties of the wave functions of quantum mechanical systems which are assumed to be so complicated that statistical considerations can be applied to them." This quantity $\rho_W(E)$ characterizes the spreading of eigenstates over the levels of an unperturbed system (e.g., in the absence of interaction between particles), and allows us to estimate how many of these unperturbed states contribute to the real wave function. Generally $\rho_W(E)$ has a Breit-Wigner distribution with Lorentzian shape of width Γ which determines the energy spreading over unperturbed states. This concept has been shown since then to be very important in a wide range of physical problems, from nuclear physics and many-electron atoms and molecules to condensed matter.

The study of such complex systems has been successfully performed through the theory of random matrices (see, for example, [2]). Very often the physics of such systems determines some preferential basis in which the Hamiltonian matrix has large diagonal matrix elements, while the nondiagonal elements corresponding to transitions between the basis states are relatively small. The investigation of random matrices of this type has been started only recently [3–6]. It has been shown that the eigenstates of such superimposed band random matrices (SBRM) are spread over the basis states according to the Breit-Wigner distribution [6]; this has been also confirmed analytically through the supersymmetry approach [7,8]. This spreading determines the number of unperturbed states contributing to a given eigenstate, which can be measured through the inverse participation ratio (IPR) $\xi = 1/\sum_i |a_i|^4$ [6–8]. Here a_i are probability amplitudes in unperturbed states. The width Γ fixes an energy scale at which the level statistics, for example, the number variance $\Sigma_2(E)$, changes behavior from the Wigner-Dyson to the Poisson case [9]. It has been also shown that the Breit-Wigner distribution appears in the case of sparse random matrices with preferential basis [10].

While the properties of the Breit-Wigner distribution are well understood in random matrix models, the problem of real interacting finite many-body fermionic systems was much less investigated. Indeed, in the latter case

the nature of the two-body interaction should be taken into account, since it gives certain restrictions on the structure of matrix elements. A very convenient model to investigate this kind of problem has been introduced some time ago in [11,12]. This model consists of n fermions distributed over m energy orbitals, coupled by a random two-body interaction. Recently this model attracted renewed attention since it was understood that it correctly describes the statistical properties of real physical systems such as the Ce atom and the Si nucleus [13,14]. One of the main advantages of this model is that it takes into account the two-body nature of the interaction and allows us to investigate the dependence of various quantities on the interaction strength U. This property is rather important since the variation of the Breit-Wigner width with respect to U and excitation energy δE counted from the Fermi level has not been yet clearly understood. Indeed, due to the two-body nature of the interaction, only a small fraction of the multiparticle states is coupled by direct transitions. As a result, contrary to common lore [13–15], the exponential growth of the multiparticle density of states ρ_n with the number of particles n and the excitation energy δE does not imply that an exponentially small interaction leads to level mixing [16]. In a similar way this exponential growth of ρ_n does not lead to an exponential growth of the width Γ. This fact has been known in nuclear physics for some time [2,17]; however, the precise dependence of Γ on δE has not been determined up to now. The dependence of Γ on U is also not obvious, due to the absence of direct coupling between the majority of the multiparticle states. Different types of power-law dependence have been recently proposed [18,19] but a definite expression for Γ is still elusive. A similar situation exists for the IPR ξ in the basis of noninteracting eigenstates which has been studied extensively very recently [20–24]. In this paper, we address these problems and determine the dependence of Γ and ξ on the parameters above. We show that these two quantities are directly related. Surprisingly ξ can be arbitrarily large at the critical interaction strength U_c [16] where the crossover in level spacing statistics $P(s)$ between the Poisson and Wigner-Dyson distributions takes place.

To investigate these properties we choose the two-body random interaction model (TBRIM) described above and studied recently in [13,16]. In this model n fermions are located on m orbitals with one-particle energies $\epsilon_{m'}$ randomly chosen in the interval $[0, m]$ so that the average one-particle level spacing is $\Delta = 1$. The multiparticle states are distributed from the ground state $E_g \approx n^2\Delta/2$ to the maximal energy $E_t \approx mn\Delta - E_g$. These states are coupled by two-body random matrix elements, varying in the interval $[-U, U]$. Because of the two-body nature of the interaction, a given multiparticle state is only coupled to $K = 1 + n(m - n) + n(n - 1)(m - n)(m - n - 1)/4$ other states in an energy interval $B = 2m - 4$. This K is much smaller than the total number of states $N = m!/n!(m - n)!$. The density of directly coupled states $\rho_c = K/B \approx mn^2/8$ is therefore much smaller than the total density $\rho_n \approx N/(E_t - E_g)$. According to the results in [16], a crossover for $P(s)$ from Poisson to Wigner-Dyson statistics takes place at a critical interaction strength $U_c = C/\rho_c$ with $C \approx 0.58$. A similar border was also discussed in [23]. The precise value of U_c [16] was determined by the condition that $\eta = \int_0^{s_0}[P(s) - P_{WD}(s)]\,ds/\int_0^{s_0}[P_P(s) - P_{WD}(s)]\,ds = 0.3$. Here $P_P(s)$ and $P_{WD}(s)$ are the Poisson and the Wigner-Dyson distributions, respectively, and $s_0 = 0.4729\ldots$ is their intersection point. Physically this crossover happens when the coupling matrix elements become comparable to the energy spacings between directly coupled states [16,18,19]. A similar condition determines the metal-insulator transition in the Anderson model, where also the level statistics $P(s)$ changes from the Poisson distribution to the Wigner-Dyson one [25]. However, the TBRIM case differs from the Anderson model where at large system size η can take only three values $\eta = 1$ (localized), $\eta \approx 0.22$ (critical), $\eta = 0$ (delocalized), while in the TBRIM η varies smoothly near U_c [16]. Physically this difference comes from the fact that in the Anderson model the number of coupled neighbors is much smaller than the linear system size, while in the TBRIM this number is of the order of the number of states m in one of n directions associated to each particle.

While the value of U_c has been determined [16], the properties of eigenstates as a function of the interaction remained unclear. To understand these properties in the TBRIM we studied the local density of states $\rho_W(E)$ in the basis of noninteracting multiparticle states. The data were obtained for the states near the middle of the spectrum ($\pm 25\%$ from the center). The total statistics for ρ_W was kept around 10^6. We checked that $\rho_W(E)$ has a Breit-Wigner shape and analyzed the dependence of its width Γ on U. The numerical data for TBRIM clearly demonstrate the relation $\Gamma \propto U^2$ which continues up to large U values where a saturation takes place (Fig. 1). To check this dependence for larger system sizes, we investigated a slightly different model, obtained from the TBRIM by restricting ourselves to states in an energy
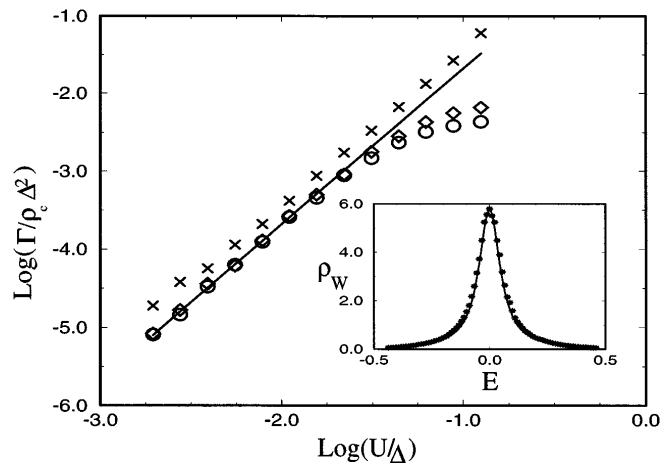

FIG. 1. Dependence of the rescaled Breit-Wigner width $\Gamma/\rho_c\Delta^2$ on U/Δ: TBRIM data for $n = 3, m = 17$ ($\times$); LM data for $n = 3, m = 130$ ($\bigcirc$) and $n = 4, m = 60$ ($\diamond$). The full line shows the theoretical estimate (1). Inset gives an example of $\rho_W(E)$ for LM ($*$) with the Breit-Wigner fit ($\Gamma = 0.12$) for $n = 3, m = 130, U = 0.022$ when (1) gives $\Gamma = 0.125$. Logarithms are decimal in Figs. 1–3.

layer of width Δ near total energy $E = m\Delta$. Such an approximation is physically reasonable provided $\Gamma \ll \Delta$. Indeed, in this case the transitions to states outside the layer do not influence the properties of eigenstates. We choose the layer to be defined by $\sum_{i=1}^n m_i' = m$. The transition matrix elements between these states were taken from the TBRIM, and the diagonal elements coming from one-particle energies $\epsilon_{m'}$ were randomly chosen in $[(m - 1/2)\Delta, (m + 1/2)\Delta]$. The layer model (LM) defined in this way retains the main physical properties of the TBRIM but allows us to study systems with a much larger number of orbitals m. For $n = 3$ the system size of LM is $\tilde{N} \approx m^2/12$ and for $n = 4$, $\tilde{N} \approx m^3/200$. This allowed us to span m values up to $m = 130$ ($n = 3, N \approx 3.6\ 10^5$) and $m = 60$ ($n = 4, N \approx 4.9\ 10^6$), which are much larger than the values reached in [13,16,22]. The multiparticle density in the LM is $\rho_n = \tilde{N}/\Delta$ while ρ_c was determined numerically. The data for LM (Fig. 1), similarly to the TBRIM case, also demonstrate the dependence $\Gamma \propto U^2$ and show in addition that $\Gamma \propto \rho_c$.

According to the data of Fig. 1 the width Γ is given by the Fermi golden rule:

$$\Gamma = 2\pi\langle U^2\rangle\rho_c = \frac{2\pi}{3}U^2\rho_c, \qquad (1)$$

where $\langle\cdots\rangle$ means the averaging. We attribute the small difference between the LM and TBRIM cases to the fact that in the latter the density ρ_c slightly depends on the energy counted from the Fermi level, while we used its average value. For the LM this variation is smaller and therefore the agreement is better.

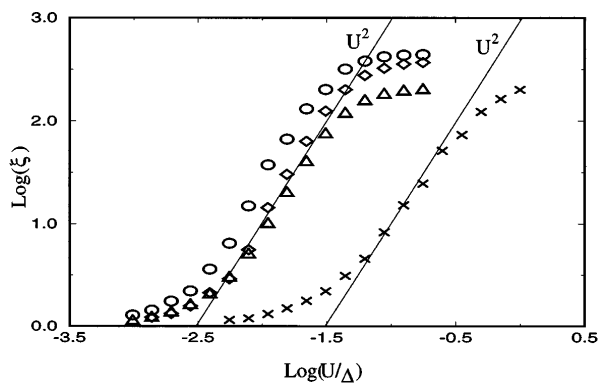The expression (1) for Γ does not depend on the multiparticle density of states ρ_n and we expect that

FIG. 2.   Dependence of the IPR ξ on U/Δ: TBRIM data for $n = 3, m = 17, U_c/\Delta = 0.055$ ($\times$); LM data for $n = 3, m = 130, U_c/\Delta = 0.0049$ ($\bigcirc$), $n = 3, m = 90, U_c/\Delta = 0.0075$ ($\triangle$), and $n = 4, m = 60, U_c/\Delta = 0.0085$ ($\diamond$).   Straight lines show dependence $\xi \propto U^2$.

for $U > U_c$ an eigenstate is spread over all unperturbed states in the energy interval Γ. In this regime level mixing goes up to level spacing $\Delta_n = 1/\rho_n \ll 1/\rho_c$ leading to

$$\xi \approx \Gamma\rho_n \approx 2U^2\rho_c\rho_n. \tag{2}$$

The numerical factor was taken in analogy with the SBRM case where $\xi \approx \Gamma\rho$ [7,8]. To check this theoretical estimate we computed ξ for both TBRIM and LM. The numerical data displayed in Fig. 2 show clearly the U^2 dependence for sufficiently strong U. At very large U the growth of ξ is replaced by a saturation due to the finite size of the system. The data shown in Fig. 3 demonstrate that $\xi \propto \rho_c\rho_n$, in agreement with (2).

Without any fitting parameters these numerical results definitely confirm the estimate (2) for $U > U_c$. It is interesting to check if it remains valid close to the critical value U_c. If so, then the IPR at U_c for $n \gg 1$ will contain exponentially many states $\xi_c = \xi(U = U_c) \sim \rho_n/\rho_c \gg$
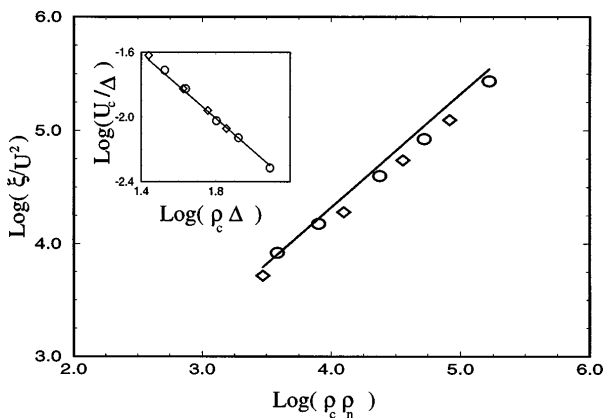

FIG. 3.   Dependence of the rescaled IPR ξ/U^2 on $\rho_c\rho_n$: LM data for $n = 3$ and $40 \leq m \leq 130$ ($\bigcirc$); $n = 4$ and $30 \leq m \leq 60$ ($\diamond$). The straight line gives theory (2). Inset shows U_c/Δ vs $\rho_c\Delta$ in log-log scale for the same parameters; the straight line is the fit $U_c = 0.62/\rho_c$.

1. We studied this behavior in both TBRIM and LM. In the latter case, we checked that U_c defined by the condition $\eta(U_c) = 0.3$ also follows the relation $U_c = C/\rho_c$, with $C = 0.62$ being very close to the TBRIM value [16] (see inset in Fig. 3). This fact once more confirms that indeed the LM retains the physical properties of TBRIM. The data of Fig. 4 for ξ_c confirm that $\xi_c \sim \rho_n/\rho_c$ in both TBRIM and LM. In LM the proportionality factor $\tilde{C}$ is about 3 times smaller ($\tilde{C} \approx \rho_c\xi_c/\rho_n \approx 0.25$ in Fig. 4) than its value given by (2) at $U = U_c$ ($\tilde{C} \approx 0.8$). This indicates a change of eigenstate properties near U_c. The difference of $\tilde{C}$ values for LM and TBRIM should be attributed to a stronger variation of the densities ρ_n, ρ_c with energy in the TBRIM. This variation was not taken into account in the expressions for ρ_n, ρ_c in the TBRIM where we used their averaged values.

The data of Fig. 4 definitely show that at $U = U_c$ the IPR grows proportionally to the multiparticle density ρ_n and, therefore, it is exponentially large for $n \gg 1$. This fact leads to the apparently surprising conclusion that for $U < U_c$ the eigenstates are composed of a huge number of noninteracting eigenstates but $P(s)$ is still close to the Poisson distribution. A similar situation is known to exist for quantum systems whose classical dynamics corresponds to the Kolmogorov-Arnold-Moser regime. In this case, the coupling between different modes strongly deforms the unperturbed tori, but does not destroy the integrals of motion and the corresponding quantum numbers. Generally such deformation gives a spreading over many unperturbed eigenstates, without real mixing of energy levels [26]. The mixing and Wigner-Dyson statistics for $P(s)$ appear only after the transition to chaos, which in our case corresponds to the situation when the physical frequency $1/\rho_c$ becomes comparable to the
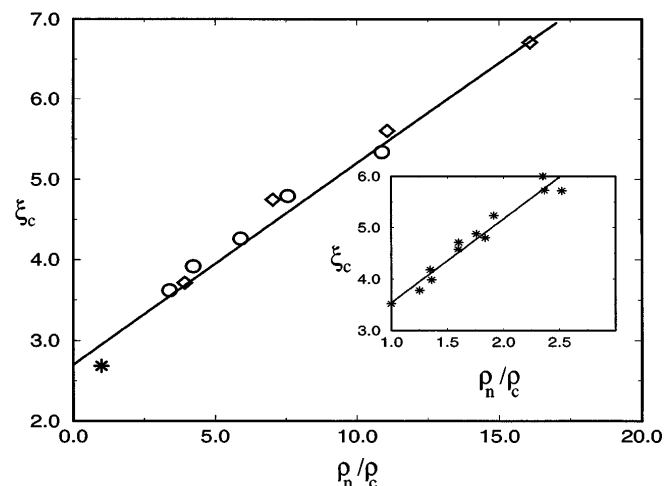

FIG. 4.   Dependence of the IPR ξ_c at U_c on ρ_n/ρ_c: LM data for $n = 2$ and $m = 800$ ($*$); $n = 3$ and $40 \leq m \leq 130$ ($\bigcirc$); $n = 4$ and $30 \leq m \leq 60$ ($\diamond$). The straight line gives the fit $\xi_c = 0.25\rho_n/\rho_c + 2.7$. Inset shows the same plot for the TBRIM for $n = 2, m = 30$; $3 \leq n \leq 6$ and $10 \leq m \leq 21$ ($*$). The straight line is the fit $\xi_c = 1.63\rho_n/\rho_c + 1.91$.

interaction-induced interstate transition rate Γ. A similar phenomenon takes place near the Anderson transition [25], but there ξ becomes infinite at/above the transition.

We have so far discussed the case of highly excited states far from the Fermi level, where ρ_n and ρ_c are not very sensitive to energy variation. This is not true near the Fermi energy $\epsilon_F \approx n\Delta$, where the dependence on excitation energy $\delta E = E - E_g$ should be taken into account. At temperature T only $\delta n \sim T/\Delta$ particles effectively interact near the Fermi level so that $\delta E \sim T\delta n \sim T^2/\Delta$. Since near ϵ_F the density $\rho_c \sim \rho_2(\delta n)^2 \sim (\delta n)^3/\Delta$ [16], we obtain

$$\Gamma \sim \frac{U^2}{\Delta}\left(\frac{\delta E}{\Delta}\right)^{3/2} \sim \frac{U^2}{\Delta}\left(\frac{T}{\Delta}\right)^2 \delta n\,;$$

$$\xi \sim \Gamma\rho_n \sim \left(\frac{U}{\Delta}\right)^2\left(\frac{\delta E}{\Delta}\right)^{1/2} \exp\left[2\left(\frac{\pi^2\delta E}{6\Delta}\right)^{1/2}\right].$$

(3)

Here we used the known dependence of ρ_n on δE from [27] and assumed that $\delta E > \delta E_{ch} \approx \Delta(\Delta/U)^{2/3}$ [$T > T_{ch} \approx \Delta(\Delta/U)^{1/3}$] so that the system is thermalized due to internal interaction ($U > U_c$) [16]. The last expression for Γ has a simple meaning. Indeed, Γ is the total spread width for δn effectively interacting particles. Therefore, the partial width $\Gamma_D \sim \Gamma/\delta n$ is the usual quasiparticle decay rate which in agreement with the theory of Landau Fermi liquid is proportional to T^2. At the quantum chaos border $\delta E = \delta E_{ch}$, when the crossover to the Wigner-Dyson statistics takes place [16], the IPR becomes exponentially large $\xi_c \sim (T_{ch}/\Delta)\exp(2.6T_{ch}/\Delta)$. We note that in the Landau Fermi liquid theory quasiparticles are well defined if $\Gamma_D < T[T < T_L = \Delta(\Delta/U)^2]$. In this regime the level statistics $P(s)$ can be as in chaotic [$P(s) = P_{WD}(s)$ for $T_{ch} < T < T_L$] or integrable [$P(s) = P_P(s)$ for $T < T_{ch} < T_L$] systems.

In conclusion, our results allowed us to understand the eigenstate properties in finite Fermi systems with interparticle interaction $U \gtrsim U_c \sim 1/\rho_c$ and excitation energy $\delta E \gtrsim \delta E_{ch}$. Qualitatively, these properties are similar and opposite to the recent expectations [22] and [13–15,18–21], correspondingly. Further investigations are required for $U < U_c$, where there are indications for another dependence of the IPR on system parameters [23].

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