Rotating Solitons and Nonrotating, Nonstatic Black Holes

O. Brodbeck, M. Heusler, N. Straumann, and M. Volkov
Institute for Theoretical Physics, University of Zurich, CH-8057 Zurich, Switzerland
(Received 29 July 1997)

It is shown that the non-Abelian black hole solutions have stationary generalizations which are parametrized by their angular momentum and electric Yang-Mills charge. In particular, there exists a nonstatic class of stationary black holes with vanishing angular momentum. It is also argued that the particlelike Bartnik-McKinnon solutions admit slowly rotating, globally regular excitations. In agreement with the non-Abelian version of the staticity theorem, these nonstatic soliton excitations carry electric charge, although their nonrotating limit is neutral. [S0031-9007(97)04637-1]

PACS numbers: 04.70.Bw

In recent years it has become obvious that a variety of well-known, and rather intuitive features of self-gravitating Maxwell fields are not shared by *non*-Abelian gauge fields. In particular, and in contrast to the Abelian situation, self-gravitating Yang-Mills (YM) fields can form particle-like configurations [1]. Moreover, the Einstein-Yang-Mills (EYM) equations also admit black hole solutions which are not uniquely characterized by their mass, angular momentum, and YM charges [2]. Hence, the celebrated uniqueness theorem for electrovacuum black hole spacetimes [3] ceases to exist for EYM systems. In fact, not even partial results of the no-hair theorem can be restored in the non-Abelian case [4]: In addition to the circumstance that spherically symmetric black holes are, in general, no longer characterized by their mass and charges, static black holes need not even be spherically symmetric [5,6]. Moreover, we shall show that there exist black hole spacetimes with vanishing angular momentum which are, however, *not* static.

The new results presented in this Letter are based on our previous investigations [7,8]. In [7] we have shown that non-Abelian black holes always have rotating counterparts. It was also conjectured that solitons generically do not admit rotating excitations. A systematic analysis of stationary perturbations revealed that this is indeed the case, provided that the EYM system is coupled to bosonic matter fields [8]. However, as the *pure* EYM system comprises exclusively massless fields, the polynomial falloff of the background configurations allows for a more general asymptotic behavior than the one considered in [7]. Hence, in the *absence* of bosonic fields, one gains an additional degree of freedom, which gives rise to the new features described in this Letter.

More precisely, we prove the existence of slowly rotating Bartnik-McKinnon (BK) solitons [1], and establish a two-parameter family of stationary excitations of the SU(2) black hole solutions. In addition to the charged, rotating solutions found in [7], there also exists a branch of uncharged, rotating black holes, and a branch of charged black holes with *vanishing* angular momentum. As these configurations are *not* static, they illustrate that the assumptions entering the non-Abelian staticity theorem [9] are optimal: According to this theorem, stationary EYM black hole solutions must be static only if they have zero angular momentum *and* vanishing electric YM charge. The new solutions demonstrate that the vanishing of the electric charge is, in fact, a *necessary* requirement for the configuration to be static. Moreover, the inversion of the non-Abelian staticity theorem also predicts that rotating excitations of the BK solitons must be charged.

Although it is, by now, mathematically clarified why slow rotations of EYM solitons are only possible in the *absence* of bosonic fields, we still lack a deeper physical understanding of this surprising fact. The authors of this Letter could not agree on any of the heuristic proposals which came up in the discussions.

Stationary perturbations.—We start by briefly recalling that the stationary perturbations of static EYM configurations are governed by a self-adjoint system of equations for a set of gauge invariant scalar amplitudes (see [8] for details). A stationary EYM configuration (with Killing field ∂_t, say) is described in terms of a stationary metric, $\boldsymbol{g}$, and a stationary non-Abelian gauge potential, A,

$$\boldsymbol{g} = -\sigma(dt + a)^2 + \sigma^{-1}\overline{\boldsymbol{g}}\,, \tag{1}$$

$$A = \phi(dt + a) + \overline{A}\,. \tag{2}$$

Here, σ and $a = a_i dx^i$ are a scalar field and a one-form on the three-dimensional (Riemannian) orbit space with metric $\overline{\boldsymbol{g}}$, respectively, and so are the Lie algebra valued quantities ϕ and $\overline{A}$, describing the electric and the magnetic part of the YM field. As we are interested in *perturbations of static, purely magnetic* configurations, both the electric potential and the off-diagonal part of the metric vanish for the unperturbed solutions, that is, $\phi \equiv \delta\phi$ and $a \equiv \delta a$.

Using the Kaluza-Klein reduction of the EYM action, we have shown in [8] that the nonstatic perturbations, δa and $\delta\phi$, decouple from the remaining metric and matter perturbations. Moreover, in first order perturbation theory, the latter do not contribute to the angular momentum. The rotational excitations of a static, purely magnetic EYM spacetime are, therefore, governed by the linearized field equations for the metric perturbation δa and the electric YM perturbation $\delta\phi$ [7].

 0031-9007/97/79(22)/4310(4)$10.00

In order to obtain a self-adjoint form of the perturbation equations, it is necessary to pass from δa to the linearized *twist potential*, $\delta\chi$, defined by

$$\delta\chi_{,k} = \varepsilon_{kij}\sqrt{\bar{g}}\left(\frac{\sigma^2}{2}\, d\delta a + 4\sigma \mathrm{Tr}\{\overline{F}\delta\phi\}\right)^{ij}. \tag{3}$$

Here, $\overline{F}$ is the field strength with respect to the magnetic potential $\overline{A}$, and the spatial indices are raised with the three-dimensional metric $\bar{\boldsymbol{g}}$. By virtue of this definition, the equations governing the nonstatic, stationary perturbations of the EYM system can, eventually, be cast into a formally *self-adjoint* system for the *gauge invariant scalar* quantities $\delta\chi$ and $\delta\phi$ [8]. (The existence of a generalized twist potential for the stationary EYM system follows from the fact that a enters the effective action only via the "field-strength" da; see [4,8] for details.)

Since the static background solutions under consideration are spherically symmetric, one can perform a multipole expansion of the perturbation amplitudes $\delta\chi$ and $\delta\phi$. Before doing so, we recall that the background metric, $\boldsymbol{g}_{BG} = -\sigma dt^2 + \sigma^{-1}\bar{\boldsymbol{g}}$ is parametrized in standard Schwarzschild coordinates by $\sigma(r)$ and $N(r)$, and the purely magnetic background gauge potential, $A_{BG} = \overline{A}$, is given in terms of a radial function $w(r)$:

$$\sigma^{-1}\bar{\boldsymbol{g}} = N^{-1}dr^2 + r^2 d\Omega^2, \tag{4}$$

$$\overline{A} = (1 - w)(\tau_\vartheta \sin\vartheta\, d\varphi - \tau_\varphi d\vartheta), \tag{5}$$

where τ_ϑ, τ_φ, and τ_r are the spherical generators of SU(2), normalized such that $[\tau_\vartheta, \tau_\varphi] = \tau_r$.

The stationary perturbations $\delta\chi$ and $\delta\phi$ can now be expanded in terms of spherical "isospin" harmonics. It turns out that all axisymmetric perturbations which give rise to rotational excitations belong to the sector with total angular momentum $j = 1$ [7]. The perturbations $\delta\chi$ and $\delta\phi$ are determined by three scalar amplitudes, $\xi_1(r)$, $\xi_2(r)$, and $\xi_3(r)$ (see [8] for details),

$$\delta\chi = 2\xi_1\cos\vartheta, \qquad \delta\phi = \xi_2\tau_r\cos\vartheta - \frac{\xi_3}{\sqrt{2}}\tau_\vartheta\sin\vartheta\,. \tag{6}$$

Using these expansions, the perturbation equations finally assume the form of a standard Sturm-Liouville equation for the three component real vector $\xi = (\xi_1, \xi_2, \xi_3)$. One finds

$$\left(-\frac{d}{dr}r^2\boldsymbol{A}\frac{d}{dr} + \boldsymbol{B}\frac{d}{dr} - \frac{d}{dr}\boldsymbol{B}^T + \boldsymbol{L} + \boldsymbol{P}\right)\xi = 0\,, \tag{7}$$

where the 3×3 matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{L}$, and $\boldsymbol{P}$ are given in terms of the background fields $w(r)$, $\sigma(r)$, and $N(r)$. The nonvanishing matrix elements of $\boldsymbol{A}$ and $\boldsymbol{B}$ are

$$\boldsymbol{A} = S^{-1}\mathrm{diag}(-\sigma^{-1}, 1, 1), \qquad \boldsymbol{B}_{21} = -2\sigma^{-1}(w^2 - 1)\,, \tag{8}$$

where we have introduced the metric function S, defined by $S^2 = \sigma/N$. For the "angular momentum" matrix $\boldsymbol{L}$ and the effective potential $\boldsymbol{P}$ one finds

$$\boldsymbol{L} = \frac{1}{NS}\begin{pmatrix} -2\sigma^{-1} & 0 & 0 \\ 0 & 2(w^2+1) & -2\sqrt{2}\,w \\ 0 & -2\sqrt{2}\,w & (w^2+1)\end{pmatrix}, \tag{9}$$

$$\boldsymbol{P} = -\frac{2}{\sigma}\begin{pmatrix} 0 & 0 & \sqrt{2}\,w' \\ 0 & 2Sr^{-2}(w^2-1)^2 & 0 \\ \sqrt{2}\,w' & 0 & 2NSw'^2\end{pmatrix}, \tag{10}$$

where a prime denotes differentiation with respect to r.

Soliton excitations.—The existence of rotational excitations of the BK soliton solutions is established as follows: First, one observes that the Sturm-Liouville equation (7) has *regular singular* points at $r = 0$ *and* $r = \infty$. This is seen by writing the perturbation equations as a six-dimensional system of first order equations, and by using the behavior of the background configurations in the vicinities of the origin and infinity. For instance, one uses

$$w = 1 - \gamma\frac{2M}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \qquad N = 1 - \frac{2M}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \tag{11}$$

and $S = 1 + \mathcal{O}(r^{-4})$, to conclude that $r = \infty$ is a regular singular point. This is, in fact, a peculiarity of the *pure* EYM system, for which the *polynomial* decay of the background fields implies that all perturbations are *massless.* (Here, M denotes the total mass and γ is a parameter characterizing the background configuration.) Taking advantage of the expansions (11) shows that the perturbation equations decouple in leading *and* in next-to-leading order. In leading order one finds a four-dimensional family of asymptotically acceptable solutions, behaving like $r^{-\lambda}$, with $\lambda = 0, 1, 2, 3$. Following the standard theory, it remains to verify that the fundamental solution belonging to $\lambda = 0$ does not exhibit logarithmic terms in next-to-leading order. In fact, it turns out that this is the case for all non-negative eigenvalues. Hence, one ends up with a *four*-dimensional system of asymptotically well-behaved local solutions:

$$\xi = \left(c_0 + \frac{c_1}{r}\right)\left[\mathbf{e}_1 + \mathcal{O}\left(\frac{\ln(r)}{r^2}\right)\right] + \frac{c_2}{r^2}\left[\mathbf{e}_2 + \mathcal{O}\left(\frac{1}{r^2}\right)\right] + \frac{c_3}{r^3}\left[\left(1 + (1-\gamma)\frac{2M}{r}\right)\mathbf{e}_3 + \mathcal{O}\left(\frac{1}{r^2}\right)\right], \tag{12}$$

where $\mathbf{e}_1 = (0, 1, \sqrt{2})$, $\mathbf{e}_2 = (1, 0, 0)$, and $\mathbf{e}_3 = (0, \sqrt{2}, -1)$. In a similar way one obtains a *three*-dimensional system of admissible solutions in the vicinity of the origin. Since the BK background solutions are continuous and regular for $0 < r < \infty$, and since the perturbation equations are linear, the local solutions in the vicinity of $r = 0$ and $r = \infty$ admit extensions to the semiopen intervals $[0, \infty)$ and $(0, \infty]$, respectively. As the total solution space is six dimensional, the intersection of the regular solution subspaces is (at least) one dimensional. Hence, all BK soliton solutions admit stationary excitations.

Black hole excitations.—As for the black hole case, one needs to investigate the behavior of solutions in the vicinity of the horizon, defined by $N(r_H) = 0$. In leading order the six fundamental solutions behave like $(r - r_H)^\lambda$, with $\lambda = 0, 1, 2$. However, a next-to-leading order expansion shows that two (out of three) solutions belonging to $\lambda = 0$ must be rejected. Since the remaining solutions are well behaved, the subspace of acceptable solutions in the vicinity of the horizon is *four* dimensional. Again using the regularity of the background configuration for $r_H < r < \infty$ shows that stationary excitations of static EYM black holes always exist. However, in contrast to the soliton case, the rotating black hole configurations are characterized by *two* parameters, rather than only one. Hence, the additional degree of freedom at the horizon implies that the intersection of the solution subspaces is now (at least) *two* dimensional.

In order to offer an interpretation of the parameters characterizing the soliton and black hole excitations, we consider the local electric YM charge and the local Komar angular momentum, defined by flux integrals over a two-sphere with radius r:

$$\tau_z Q(r) = \frac{1}{4\pi} \int * F = \frac{\tau_z}{3S}[r^2(\xi_2 + \sqrt{2}\,\xi_3)' + 2w'\beta],$$
$$J(r) = \frac{1}{16\pi} \int * (d\boldsymbol{g}_{\varphi\mu} \wedge dx^\mu) = -\frac{r^4}{6S}\left(\frac{\beta}{r^2}\right)',$$

where β parametrizes the metric perturbation, $\sigma\delta a \equiv \beta(r)\sin^2\vartheta\, d\varphi$ [see Eq. (1)]. By virtue of the harmonic expansions (6) and the definition (3) of the twist potential $\delta\chi$, one obtains an expression for β in terms of the perturbation amplitudes ξ_i,

$$\beta = 2(w^2 - 1)\xi_2 + S^{-1} r^2 \xi_1' . \tag{13}$$

The electric YM charge, Q, and the Komar angular momentum, J, are obtained from the above local expressions in the limit $r \to \infty$, where the asymptotic expansion (12) yields $c_1 = -Q$ and $c_2 = -(J + 4\gamma M c_0)$. The leading two terms in the asymptotic expansion of the electric potential $\delta\phi$ and the metric one-form δa are, therefore (with $q = Q + Mc_0(5\gamma - 3)/2$),

$$\delta\phi = \left(c_0 - \frac{Q}{r}\right)\tau_z ,$$
$$\sigma\delta a = 2\left(\frac{J}{r} + \gamma\frac{4Mq}{r^2}\right)\sin^2\vartheta\, d\varphi . \tag{14}$$

For perturbations of a *Schwarzschild* background, the above expressions are, in fact, the exact solutions of the perturbation equations, where the second term in δa is absent, since $\gamma = 0$ in this case. (Note that the Schwarzschild background solution is given by $w = 1$, $S = 1$, $\sigma = N = 1 - 2M/r$.) As c_0 does not enter the *Abelian* field strength, $F = d\delta\phi \wedge dt$, it has no physical significance and may, as usual, be set equal to zero. Hence, as expected, the stationary excitations of the Schwarzschild solution are linearized Kerr-Newman solutions, parametrized by their charge Q and their angular momentum J [7]. In particular, it is consistent to consider perturbations with either $Q = 0$ (Kerr) or $J = 0$ (Reissner-Nordström).

Returning to the stationary excitations of the non-Abelian black holes, we first emphasize that the constant c_0 now has decisive physical consequences. In fact, by virtue of the covariant derivative, c_0 enters the asymptotic expression for the field strength. (It does, however, not show up in the expression for Q, since the corresponding two-form in the formula for $*F$ is not proportional to the volume-form of the two-sphere.) As we have argued above, one obtains a two-dimensional family of excitations in the black hole case, provided that the nontrivial asymptotic degree of freedom, c_0, is taken into account. Hence one can, in particular, consider solutions with either $Q = 0$, $J = 0$, or, as in [7], $c_0 = 0$.

We start with the uncharged excitations of EYM black holes, $Q = 0$. As in the Abelian case, these have a nonstatic metric, $\delta a \neq 0$, and are rotating, $J \neq 0$. However, despite the fact that the electric YM charge vanishes, there now arises a nonvanishing electric YM field, $E = d\delta\phi + [\overline{A}, \delta\phi]$. Asymptotically, this becomes

$$E = \tau_z \frac{Q}{r^2}\, dr + 2\gamma M \frac{c_0}{r}\,(\tau_r d\cos\vartheta - \cos\vartheta\, d\tau_r)\,, \tag{15}$$

which vanishes for $Q = 0$ only in the Abelian case (since then $w = 1$, i.e., $\gamma = 0$). (As already mentioned, the c_0 term is tangential to the two-sphere and does, therefore, not contribute to the electric YM charge. It is also not hard to verify that the contributions of this term to the total energy and to the action are finite.)

Even more interesting is the class of stationary excitations with $J = 0$. Whereas in the Abelian case $J = 0$ implies $\delta a = 0$, this is no more true for perturbations of static EYM black holes: Despite the fact that the angular momentum vanishes, the perturbed metric is not static, as is already seen from the asymptotic behavior (14). (Again, this effect is proportional to γ, which vanishes for a Schwarzschild background.) This shows that there do exist EYM black hole solutions with a nonstatic domain of outer communications and vanishing angular momentum. It is worthwhile noticing that the local angular momentum, $J(r)$, does not vanish when evaluated for *finite* values of r, in particular, for $r = r_H$; see Fig. 1. Hence, these black holes have a rotating horizon, $J(r_H) \neq 0$, although they are nonrotating in the sense that $J = 0$. (In contrast to this, a Kerr-Newman black hole with $J = 0$ also has $J(r_H) = 0$, since both quantities are proportional to the Kerr rotation parameter.) Numerical results for $J(r)$ and $Q(r)$ are shown in Fig. 1. We also expect that these black holes have an ergosphere (that is, a region in the domain of outer communications where the Killing field ∂_t becomes spacelike). This does, however, not show up in the lowest order perturbation theory, since the metric field σ is a background quantity within this approximation.

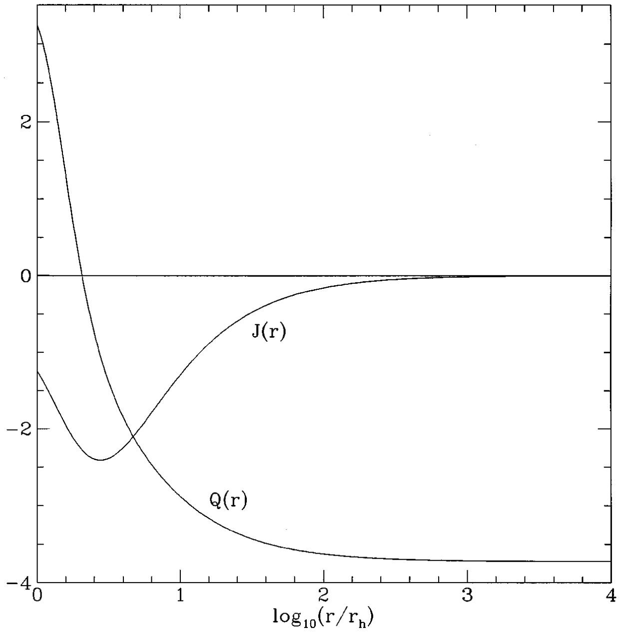

FIG. 1. The local charge $Q(r)$ and the local Komar angular momentum $J(r)$ for the nonrotating, nonstatic excitation of the non-Abelian background black hole with $n = 1$, $r_H = 1$.

The rotating *solitons* are characterized by one, rather than two parameters. This is due to the fact that the solution space at the origin has one dimension less than the solution space at the horizon. Hence, the charge Q and the angular momentum J are not independent infinitesimal hair any longer. Rotating stationary excitations of the BK solution are, therefore, electrically charged.

The Abelian staticity conjecture [10] asserts that stationary, nonrotating black hole solutions to the Einstein-Maxwell equations are static. In 1992, Sudarsky and Wald were able to prove this long-standing conjecture and, in addition, also established a non-Abelian version of the theorem [9]. Their result shows that

$$\Omega_H J - \mathrm{Tr}\{\phi_\infty Q\} = 0 \Rightarrow a \equiv 0, \quad \text{and } E \equiv 0\,, \tag{16}$$

where Ω_H is the angular velocity of the horizon, E is the electric YM field, and a is the nonstatic part of the metric, defined in Eq. (1). While this proves that nonrotating, *uncharged* EYM black holes are indeed static, it does not allow the same conclusion in the presence of electric YM charges. The class of stationary, nonstatic black holes discussed above illustrates that $Q = 0$ is not only a sufficient, but indeed a necessary condition for metric staticity, $a = 0$. Moreover, theorem (16) provides an explanation for the charge-up of rotating *solitons:* Since the first term is not present for soliton configurations, one concludes that nonstatic excitations ($a \neq 0$) must have nonvanishing electric YM charge. In addition, the theorem also implies that these solutions can exist only if ϕ_∞ does not vanish, which reflects the crucial importance of the constant term, c_0, in the asymptotic expansion (12).

In conclusion, we have investigated stationary perturbations of static soliton and black hole solutions to the pure EYM equations. In contrast to boson stars [11] or soliton configurations with Higgs fields [8], the BK solitons do admit rotating excitations with continuous angular momentum. We have argued that this particular feature of the pure EYM system is due to the slow (polynomial) decay of the static background configurations. The stationary excitations of EYM black hole solutions form a two-parameter family. In particular, we have presented a class of nonstatic black hole spacetimes with vanishing total angular momentum. Both the existence of a second branch of black holes and the charge-up of solitons due to rotation are typical non-Abelian features of the pure EYM system. While we have shown earlier [4] that the Abelian circularity theorem does not generalize to EYM systems in a straightforward manner, the solutions presented in this Letter show that the same is true for the Abelian staticity theorem: In the non-Abelian case, stationary black hole spacetimes with vanishing angular momentum need not be static, unless they have vanishing electric YM charges.

We have not studied the stability of the new solutions, but there is no reason to expect that the unstable modes of the BK solitons and their black hole counterparts [12] will disappear if the angular momentum and/or the YM charges are varied. The work of Ridgway and Weinberg [5], establishing the existence of *stable* solutions without spherical symmetry near the Reissner-Nordström metric is, therefore, still exceptional.

[1] R. Bartnik and J. McKinnon, Phys. Rev. Lett. **61**, 141 (1988).
[2] M. S. Volkov and D. V. Gal'tsov, JETP Lett. **50**, 345 (1989); H. P. Künzle and A. K. M. Masood-ul-Alam, J. Math. Phys. (N.Y.) **31**, 928 (1990); P. Bizon, Phys. Rev. Lett. **64**, 2644 (1990).
[3] See, e.g., M. Heusler, *Black Hole Uniqueness Theorems* (Cambridge University Press, Cambridge, England, 1996); P. T. Chruściel, in *Differential Geometry and Mathematical Physics,* edited by J. Beem and K. L. Duggal (Am. Math. Soc., Providence, 1994).
[4] M. Heusler, Helv. Phys. Acta **69**, 501 (1996).
[5] S. A. Ridgway and E. J. Weinberg, Phys. Rev. D **52**, 3440 (1995).
[6] B. Kleihaus and J. Kunz, Phys. Rev. Lett. **78**, 2527 (1997).
[7] M. S. Volkov and N. Straumann, Phys. Rev. Lett. **79**, 1428 (1997).
[8] O. Brodbeck and M. Heusler, gr-qc 9706064.
[9] D. Sudarsky and R. M. Wald, Phys. Rev. D **47**, R5209 (1993); **46**, 1453 (1992); see also P. T. Chruściel and R. M. Wald, Commun. Math. Phys. **163**, 561 (1994).
[10] B. Carter, in *Gravitation in Astrophysics,* edited by B. Carter and J. B. Hartle (Plenum, New York, 1987).
[11] Y. Kobayashi, M. Kasai, and T. Futamase, Phys. Rev. D **50**, 7721 (1994); F. D. Ryan, Phys. Rev. D **55**, 6081 (1997); S. Yoshida and Y. Eriguchi, Phys. Rev. D **56**, 762 (1997); F. Schunck and E. Mielke, in *Relativity and Scientific Computing,* edited by F. Hehl, R. Puntigam, and H. Ruder (Springer, Berlin, 1996).
[12] N. Straumann and Z. Zhou, Phys. Lett. B **237**, 353 (1990); **243**, 33 (1990).