Instability and Stretching of Vortex Lines in the Three-Dimensional Complex Ginzburg-Landau Equation

I. S. Aranson[1,2] and A. R. Bishop[3]
[1]*Department of Physics, Bar Ilan University, Ramat Gan 52900, Israel*
[2]*Argonne National Laboratory, 9700 South Cass Avenue, Argonne, Illinois 60439*
[3]*Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, New Mexico 87545*
(Received 19 May 1997)

The dynamics of curved vortex filaments is studied analytically and numerically in the framework of a three-dimensional complex Ginzburg-Landau equation (CGLE). It is proved that a straight vortex line is unstable with respect to spontaneous stretching and bending in a certain range of parameters of the CGLE, resulting in formation of persistent entangled vortex configurations. The analysis shows that the standard approach relating the velocity of the filament with the local curvature is insufficient to describe the instability and stretching of vortex lines. [S0031-9007(97)04441-4]

PACS numbers: 47.32.Cc, 05.45.+b, 47.20.Ky, 47.27.Eq

The complex Ginzburg-Landau equation (CGLE), derived some 20 years ago by Newell [1] and Kuramoto [2], has become a paradigm model for a qualitative description of weakly nonlinear oscillatory media (see for review [3]). Under appropriate scaling of the physical variables, the equation assumes the universal form

$$\partial_t A = A - (1 + ic)|A|^2 A + (1 + ib)\Delta A\,, \quad (1)$$

where A is the complex amplitude, b and c are real parameters, and $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ is a three-dimensional Laplace operator. Although the equation is formally valid only at the threshold of a supercritical Hopf bifurcation, it has been found that the CGLE often reproduces qualitatively correct phenomenology over a much wider range of the parameters. As a result, the predictions drawn from the analysis of the CGLE (mostly in one and two spatial dimensions, see, e.g., [4–6]) were recently successfully confirmed by experiments in optical and chemical systems [7,8]. Moreover, some results obtained from the CGLE (for example, symmetry breaking of spiral pairs) were instructive for interpretation of experiments in far more complicated systems of chemical waves [9] and colonies of amoebae [10,11].

Recently, the dynamics of *three-dimensional* (3D) vortex filaments has attracted substantial attention [12–16]. In the context of the 3D CGLE, Gabbay, Ott, and Guzdar [13] applied a generalization of Keener's method for a scroll vortex in reaction-diffusion systems [17]. They derived that the ring of radius R collapses in finite time according to the following evolution law:

$$\frac{dR}{dt} = -\frac{1 + b^2}{R}\,, \quad (2)$$

This result generalizes Keener's ansatz by including the curvature-induced shift of the filament's wave number and corrects the erroneous estimate of Ref. [12]. Thereby, as follows from Eq. (2), vortex filaments initially existing in the system will always shrink (if, of course, there is no bulk instability of the waves emitted by the vortex filament), and under no condition can the vortex filament expand [18].

In the Letter we show that under very general conditions and in an extensive part of the parameter space vortex filaments *expand* and result in persistent vortex configurations even if there is no bulk instability of emitted waves. The condition for the expansion of the vortex filament is simply $b > b_c(c)$. In this limit Eq. (2) is not valid, because formally higher-order corrections, omitted in Eq. (2), cause severe instability of the filament and persistent stretching. This instability is a three-dimensional manifestation of the two-dimensional core instability of spiral waves (called in Ref. [5] *acceleration instability*). It originates from breakdown of the Galilean invariance of the CGLE for any $\epsilon = 1/b \neq 0$, causing spontaneous acceleration of the spiral waves [5]. Although in 2D the instability is relatively weak, the situation is different in three dimensions. As we will show, the bending of the filament greatly enhances the instability, resulting in formation, after some transient, of a highly entangled and dense vortex configuration. The condition $b \gg 1$ is readily fulfilled for many physical and chemical systems. For example, in the context of nonlinear optics, where the CGLE can be derived from the Maxwell-Bloch equation in the good cavity limit [19], this parameter is very small: $\epsilon = 1/b \sim 10^{-4}$–$10^{-3}$. For an oscillating chemical reaction the diffusion rates of various components can be varied over a wide range by adding extra chemicals.

As a test for instability, we consider the dynamics of a weakly curved vortex filament. We apply the generalization of the method of Ref. [5] for the case of 3D vortices, and make perturbations near the 2D spiral wave solution to the CGLE. For convenience, we redefine $\mathbf{r} \to \mathbf{r}/\sqrt{b}$. Then Eq. (1) assumes the form

$$\partial_t A = A - (1 + ic)|A|^2 A + (\epsilon + i)\Delta A\,. \quad (3)$$

In the following discussion, we assume $0 < \epsilon \ll 1$ to be a small parameter. Our objective is to relate the acceleration of the vortex filament $\partial_t \mathbf{v}$ with the velocity $\mathbf{v}$ and local

 0031-9007/97/79(21)/4174(4)$10.00

curvature κ of the filament. We obtain that for $\epsilon \ll 1$ the equation of filament motion can be written as

$$\partial_t \mathbf{v} + \hat{K}[\epsilon \mathbf{v} + (1 + \epsilon^2)\kappa \mathbf{N}] = 0, \tag{4}$$

where $\mathbf{N}$ is the unit vector pointing toward the center of curvature, and $\mathbf{B}$ is the unit vector perpendicular to the filament and $\mathbf{N}$. Velocity and acceleration have correspondingly two components, $\mathbf{v} = (v_N, v_B)$, $\partial_t \mathbf{v} = (\partial_t v_N, \partial_t v_B)$. $\hat{K}$ is the friction tensor satisfying $K_{11} = K_{22}, K_{12} = -K_{21}$. Dropping the first term in Eq. (4), we reproduce the result of Ref. [13] for the collapse rate $v_N = -(1 + \epsilon^2)\kappa/\epsilon$ (because $\kappa = -1/R$ and $v_n = -\partial_t R$). However, if $K_{11} < 0$, which we will show is the case for $\epsilon < \epsilon_c(c)$ [4], the acceleration has a significant effect and leads to a stretching of the vortex filaments and the persistence of entangled vortex configurations. Moreover, the three-dimensional instability persists for (small) positive K_{11}, i.e., for $\epsilon > \epsilon_c$. For $c = 0.5$ we obtained $\epsilon_c \approx 0.1557$, whereas the stretching instability disappears at $\epsilon = 0.37$.

In order to develop the perturbation theory for a weakly curved vortex filament in 3D, we begin with the stationary one-armed isolated spiral solution to Eq. (3), which is of the form

$$A_0(r, \theta) = F(r) \exp i[\omega t \pm \theta + \psi(r)], \tag{5}$$

where (r, θ) are polar coordinates. The real functions F and ψ have the following asymptotic behavior $F(r) \to \sqrt{1 - \epsilon k_0^2}$, $\psi'(r) \to k_0$ for $r \to \infty$ and $F(r) \sim r$, $\psi'(r) \sim r$ for $r \to 0$. The wave number k_0 of the waves emitted by the spiral is determined uniquely for given ε, c [20]. For $\epsilon = 0$ one has a type of Galilean invariance, and then, in addition to the stationary spiral, there exists a family of spirals moving with arbitrary constant velocity $\mathbf{v}$ [5],

$$A(r, t) = F(r') \exp i\left[\omega' t + \theta + \psi(r') - \frac{\mathbf{r}' \cdot \mathbf{v}}{2}\right], \tag{6}$$

where $\mathbf{r}' = \mathbf{r} + \mathbf{v}t$, $\omega' = \omega + \mathbf{v}^2/4$, and the functions F, ψ are those of Eq. (5). (This invariance holds for any stationary solution.) For $\epsilon \neq 0$ the diffusion term $\sim \varepsilon \Delta A$ destroys the family and leads to acceleration or deceleration of the spiral proportional to ϵv, depending on the value of ϵ. For $\epsilon < \epsilon_c(c)$ the spiral is unstable with respect to spontaneous acceleration since $K_{11} < 0$ [5].

Let us now consider the dynamics of an almost straight vortex line in the filament-based coordinate system (see for details [13]). The position in space X is represented by local coordinates $s, \tilde{x}, \tilde{y}$, where s is the arclength along the filament, and $\mathbf{X} = \mathbf{R}(s) + \tilde{x}\mathbf{N}(s) + \tilde{y}\mathbf{B}(s)$, where $\mathbf{R}$ is the coordinate of the filament. In this basis the weakly curved filament moving with velocity $\mathbf{v}$ can be written in the form

$$A(r, t) = F(r') \exp\left[i\left(\omega' t + \theta + \psi(r') - \frac{\mathbf{r}' \cdot \mathbf{v}}{2} + \frac{\delta k_x \tilde{x}}{2}\right)\right] + W(r', \theta, s). \tag{7}$$

Here $\mathbf{r}' = \mathbf{r} + \mathbf{v}\mathbf{t}$, W is the perturbation to the spiral solution which we require to be small, and δk_x is the correction to the wave number due to curvature of the filament. In principle, this correction can be absorbed in W but it is convenient to retain this form since we can cancel part of the perturbation exactly by adjusting δk_x [13]. The perturbation procedure to derive Eq. (4) is practically identical to that of Refs. [4,5]. Substituting the ansatz Eq. (7) into the CGLE, and assuming $|\partial_t v|, \kappa \ll 1$, one obtains from Eq. (3) in first order in ϵ an inhomogeneous linear equation for the correction W, which is of the form $\hat{L}W = H$, where $\hat{L}W = [1 + i\omega - 2(1 + ic)|A_0|^2]W - (1 + ic)A_0^2 W^* + i\Delta_\perp W$, $\Delta_\perp = \partial_{\tilde{x}}^2 + \partial_{\tilde{y}}^2$. In first order the W-independent part H has two terms, $H = H_{\rm tr} + H_{\rm rot}$, where $H_{\rm tr}$ contributes to the displacement of the vortex filament, and $H_{\rm rot}$ contributes to the shift of the vortex frequency. We are interested only in $H_{\rm tr}$ (the contribution $H_{\rm rot}$ is not responsible for the instability), which is of the form

$$H_{\rm tr} = -\mathbf{r}'\partial_{\mathbf{t}}\mathbf{v}\,\frac{iA_0}{2} + i\epsilon\mathbf{v}\nabla A_0 + (\epsilon + i)(\kappa - i\delta k_x)\nabla A_0. \tag{8}$$

The derived system of equations is very close to that considered in Ref. [5] in the context of the acceleration instability of a spiral wave in 2D. The acceleration can be obtained as a result of the solvability condition requiring that W be regular at the core of the spiral and does not diverge *exponentially* at large r. (Slower, power-law divergence of W is permitted.) To impose the solvability condition a specific numerical procedure is required (see for details [4,5]). Equation (8) contains an extra term $(\epsilon + i)(\kappa - i\delta k_x)\partial_{\tilde{x}} A_0$ with respect to that considered in [5], which originates from the Laplace operator in a local basis (in the limit of small curvature κ and torsion τ): $\Delta = -\kappa\partial_{\tilde{x}} + \partial_{\tilde{x}}^2 + \partial_{\tilde{y}}^2 + \partial_s^2 + \dots$. However, additional numerical integration is not necessary to account for the effect of this term, because, as was found in Ref. [13], this term can be canceled *exactly* to first order in κ by a proper choice of $\delta k_x = -\epsilon\kappa$. After this modification, $H_{\rm tr}$ assumes the form $H_{\rm tr} = -\mathbf{r}'\partial_t \mathbf{v}\frac{iA_0}{2} + i\epsilon\overline{\mathbf{v}}\nabla A_0$, where $\overline{v}_N = v_N + (1 + \epsilon^2)\kappa/\epsilon$ and $\overline{v}_B = v_B$. It is easy to see that it now coincides with the corresponding function considered in Ref. [5]. Following the lines of analysis of Ref. [5], we observe that the relation between $\partial_t v$ and $\overline{v}$ is identical to that considered in [5], and is of the form

$$\partial_{\mathbf{t}}\mathbf{v} + \epsilon\hat{K}\overline{\mathbf{v}} = 0. \tag{9}$$

Thus we obtain Eq. (4). The coefficients of the friction tensor K_{ij} coincide with those calculated in Ref. [5] using numerical matching, and for $\epsilon < \epsilon_c$ it was found that $K_{11} < 0$, which would guarantee the instability with respect to spontaneous acceleration of the vortex filament. Since in 3D the direction of motion of the filament in general varies along the filament, this instability results in stretching of the vortex line.

In the case of the acceleration instability, $K_{11} < 0$ and the curvature has a strong destabilizing effect. Indeed, for an almost straight vortex parallel to the axis z, we can parametrize the position along the filament by the z coordinate: $(X_0(z), Y_0(z))$. Since in this limit the arclength s is close to z, the curvature correction to the velocity $\kappa \mathbf{N}$ is simply $\kappa \mathbf{N} = (\partial_z^2 X_0, \partial_z^2 Y_0)$. After simple algebra one derives the following relation for the growth rate $\lambda(k)$ of linear perturbation $X_0(z)$, $Y_0(z) \sim \exp[\lambda(k)t + ikz]$: $\lambda^2 + (K_{11} + iK_{12})(\epsilon\lambda + k^2) = 0$. For $k \gg \epsilon$ we obtain $\lambda \approx \pm\sqrt{-(K_{11} + iK_{12})}\,k \gg \epsilon$. Thus, for finite k the growth rate $\lambda(k)$ may significantly exceed the increment of the acceleration instability in 2D (corresponding to $k = 0$): $\lambda_0 = -\epsilon(K_{11} + iK_{12})$. We can expect that, as a result of such an instability, highly curved vortex filaments will be formed. Therefore, the above considered "small-curvature" approximation can be valid only for finite time. Moreover, it is naive to expect a saturation of this instability in a steady-state configuration with finite curvature. In contrast, we suggest that frequent reconnection of various parts of the filaments, formation of vortex loops, etc., will result in persistent spatiotemporal dynamics of a highly entangled vortex state.

In order to follow further development of the instability we performed numerical simulations of the 3D CGLE. We studied a system of 50^3 dimensionless units of Eq. (3) with no-flux boundary conditions. The numerical solution was performed on an R10000 SGI workstation by an implicity Crank-Nicholson algorithm. The number of grid points was 100^3. We performed simulations for $\epsilon = 0.02$ in the parameter regime away from amplitude turbulence in 2D [5]. As an initial condition we used a straight vortex line perturbed periodically or by small noise. In Fig. 1 we show the dependence of real part of the growth rate for periodic perturbation obtained by direct integration of CGLE. The result is in convincing agreement with the theoretical dependence in small k limit. Remarkably, the maximum growth rate of the 3D instability exceeds by 2 *orders of magnitude* the corresponding growth rate in 2D. The evolution of straight vortex is shown in Fig. 2. We see that the length of the vortex line grows. The dynamics seems to be very rapidly varying in time, and the line intersects itself many times, forming numerous vortex loops. The long-time dynamics show, however, some kind of saturation when a highly entangled vortex state is formed and the length of the line cannot grow any more due to repulsive interaction between closely packed line segments. The dependence of the line length on time is shown in Fig. 3. As a measure of the filament length L we used the following quantity: $L \approx S_0^{-1} \int \Theta[A_0 - |A(x,y,z)|]dx\,dy\,dz$, where Θ is a step function, $\Theta(x) = 1$ if $x > 0$ and $\Theta = 0$ otherwise. $A_0 = 0.1$ was used as a threshold value to identify the vortex. S_0 is a constant determined from the condition that for the straight line the above integral coincides with the actual length. We can identify two distinct stages of the dynamics: First, fast growth of the length; second, oscillations of the line's length around some mean value. Remarkably, we observed an increase in the amplitude of the oscillation moving through negative c values in the regime of spatiotemporal intermittency in the 2D CGLE. It is plausible that the symmetry breaking mechanism responsible for persistent intermittent behavior in the 2D system still has some importance in 3D.

Persistent entangled vortex configurations are known from numerical simulations of excitable reaction-diffusion systems [14,15,21]. However, there is also a significant difference between these two phenomena. For the

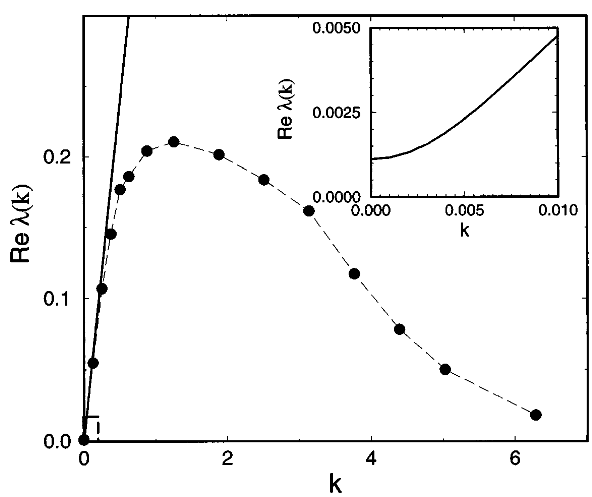

FIG. 1. The growth rate $\mathrm{Re}\lambda(k)$ as function of k for $\epsilon = 0.02$, $c = 0.1$. Solid line is the theoretical result for $k \ll 1$, dashed line with symbols the result of numerical solution of 3D CGLE. Inset: Blowup of small k region.

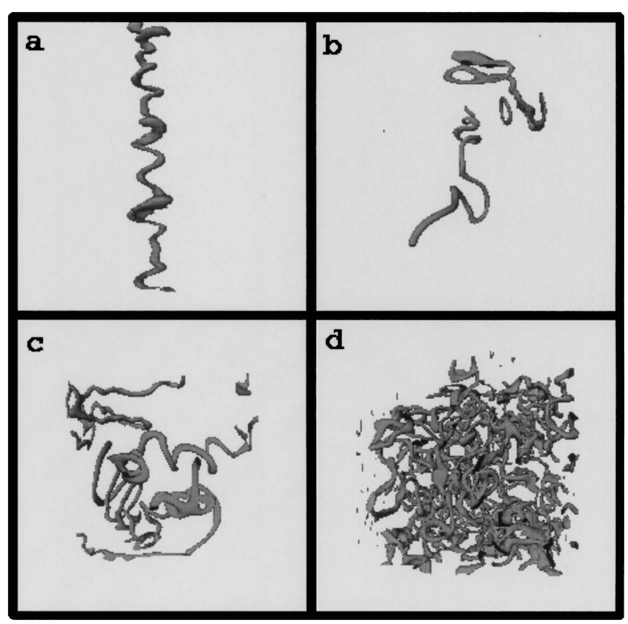

FIG. 2. Instability of a straight vortex filament. 3D isosurfaces of $|A(x,y,z)| = 0.1$ for $\epsilon = 0.02$, $c = -0.03$, shown at four times: 50 (a), 150 (b), 250 (c), and 500 (d).

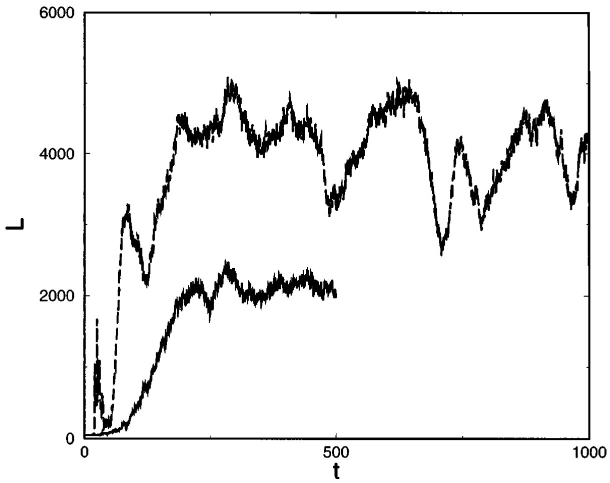

FIG. 3. The dependence of filament length L on time. Solid line corresponds to $\epsilon = 0.02$, $c = -0.03$; dashed line corresponds to $\epsilon = 0.02$, $c = -0.5$.

reaction-diffusion systems the meandering instability is typically supercritical, and may be saturated at a relatively small amplitude of oscillations. In application to 3D systems, one expects that highly curved vortex filaments will not develop, and for this reason the intersections are very seldom. This picture is consistent with the numerical simulations. Consequently, one may expect that the analysis performed on the basis of a small-curvature approximation will be valid for a very long time and can capture the actual dynamics of the filament. In the CGLE the 2D instability is subcritical and the velocity of the spiral in an infinite system can grow arbitrarily until additional topological defects nucleate in the wake of the accelerating spiral. In the 3D situation we have shown that curvature of the filament is even a destabilizing factor. Our numerical simulations show that the filament does not approach any steady state. In contrast, it shows "violent" intermittent behavior, with numerous reconnections and splitting of vortex rings. As a result, the approximate equation of motion (4) must be considered only as a test for instability rather than long-time evolution.

In conclusion, we have derived an equation of motion for the vortex filament in the CGLE. We have found that in a wide range of parameters of the CGLE the vortex filament is unstable with respect to spontaneous stretching, resulting in the formation of persistent entangled vortex configurations. This emphasizes the deficiency of previous approaches relating local filament velocity to local curvature. Our result could be verified in experiments with autocatalytic chemical reactions in gels in the regime of oscillatory instability. Also, the limit of a large dispersion $b \gg 1$ can probably be achieved by doping with additional chemicals, thus changing the relative mobility of one chemical species with respect to another. Recently, the amplitude equation governing the dynamics of an elastic rod was derived [22]. We note that our instability can be formally interpreted as a dynamics of a thin rod with *negative* elasticity. We also speculate that our results are relevant for inviscid hydrodynamics. In the limit of $b, c \rightarrow \infty$, Eq. (1) reduces to the defocusing nonlinear Schrödinger equation (NSE), which is a paradigm model for compressible inviscid hydrodynamics. Although the vortex lines are stable in the framework of the NSE, the corrections arising from the CGLE cause their destabilization and stretching.

We are grateful to A. Newell, D. Levermore, C. Doering, and R. Goldstein for illuminating discussions. This work was supported by the U.S. Department of Energy under Contracts No. W-31-109-ENG-38 (I. A.) and No. ERW-E420 (A. B.). The work of I. A. was also supported by the NSF, Office of STC under Contract No. DMR91-20000.

[1] A. C. Newell, *Envelope Equations* (American Mathematical Society, Providence, RI, 1974).
[2] Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence* (Springer-Verlag, Berlin, 1983).
[3] M. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).
[4] I. Aranson, L. Kramer, and A. Weber, Phys. Rev. E **47**, 3231 (1993); **48**, R9 (1993).
[5] I. Aranson, L. Kramer, and A. Weber, Phys. Rev. Lett. **72**, 2316 (1994).
[6] G. Huber, P. Alstrøm, and T. Bohr, Phys. Rev. Lett. **69**, 2380 (1992); H. Chatê and P. Manneville, Physica (Amsterdam) **224A**, 348 (1996).
[7] F. T. Arrecchi *et al.*, Phys. Rev. Lett. **67**, 3749 (1991).
[8] Q. Ouyang and J.-M. Flesselles, Nature (London) **379**, 6561 (1996).
[9] M. Ruiz-Villarreal, M. Gomez-Gesteira, and V. Perez-Villar, Phys. Rev. Lett. **78**, 779 (1997).
[10] I. Aranson, H. Levine, and L. Tsimring, Phys. Rev. Lett. **72**, 2561 (1994).
[11] K. J. Lee, E. C. Cox, and R. E. Goldstein, Phys. Rev. Lett. **76**, 1174 (1996).
[12] T. Frisch and S. Rica, Physica (Amsterdam) **61D**, 155 (1992).
[13] M. Gabbay, E. Ott, and P. Guzdar, Phys. Rev. Lett. **78**, 2012 (1997).
[14] P. Fenton and A. Karma, "Scroll Wave Dynamics in a Model of Cardiac Tissue with Restitution and Fiber Rotation" (to be published).
[15] A. T. Winfree, Physica (Amsterdam) **84D**, 126 (1995).
[16] A. Winfree *et al.*, Chaos **6**, 617 (1996).
[17] J. P. Keener, Physica (Amsterdam) **31D**, 269 (1988).
[18] As a vortex line we define the line singularity of the phase of complex function A.
[19] J. Lega, J. V. Moloney, and A. C. Newell, Phys. Rev. Lett. **73**, 2978 (1994).
[20] P. Hagan, SIAM J. Appl. Math. **42**, 762 (1982).
[21] V. Hadkim and A. Karma (to be published).
[22] A. Goriely and M. Tabor, Phys. Rev. Lett. **77**, 3537 (1996).