Experimental Observation of the Two-Dimensional Inverse Energy Cascade

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We report the first clear experimental evidence of a stationary two-dimensional inverse energy cascade. The experiments are performed in electromagnetically driven flows, using thin, stably stratified layers; measurements of the instantaneous velocity fields show that the fluctuations are homogeneous and isotropic in the inertial range. The energy spectrum displays the expected $k^{-5/3}$ law with a Kolmogorov constant lying in the range 5.5–7, which is consistent with the current numerical estimates. [S0031-9007(97)04567-5]

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Thirty years ago, Kraichnan [1] proposed that two-dimensional turbulence, forced at a fixed scale, should display a $k^{-5/3}$ energy spectrum, extending from the injection scale towards the large scales. This is known as the inverse energy cascade. Since then, several numerical simulations have been performed [2–5], all giving strong support to this proposal. Although the number of simulations is still rather modest, it is generally accepted that the inverse cascade is a universal phenomenon which occurs for a broad range of conditions, independently of the particular way that the energy is dissipated at large scales, and the details of the forcing. This view is supported by the fact that the Kolmogorov constant found in all the simulations, within numerical uncertainty, is the same (falling between 5.8 and 9), despite the variety of conditions of preparation of the turbulent regimes. It has also been shown [4] that, when the most energetic wave number reaches the system size, coherent vortices emerge which may eventually modify the spectral content of the turbulent regimes. This phenomenon is often referred to, following Kraichnan [1], as Bose-Einstein condensation. The inverse cascade and the condensation process were also observed in a pioneering laboratory work by Sommeria [6]. In his experiments, however, the Kolmogorov constant was found to depend on the rate at which energy is extracted at large scales. Since the measurements were performed using a line of probes, it was not possible to check whether this dependence signals a violation of the conditions under which the cascade is supposed to take place, such as the isotropy or the homogeneity of the flow, or whether it suggests that the cascade itself is not a universal process.

In the present experimental study, we are able to investigate the inverse energy cascade in a system where long runs can be performed, and for which the complete velocity and vorticity fields can be measured at all times in a completely nonintrusive way. This allows one to check the homogeneity and isotropy of the system, and to compute the spectral properties without making any assumption about the flow symmetries. In our experiments, the spectra display a clear inertial range with $k^{-5/3}$ scaling, and stationarity is achieved over time scales of the same order as the longest simulations performed on the subject [7]. Moreover, the initial transient evolution toward the $k^{-5/3}$ stationary state is observed for the first time. Additionally, the experiment provides a value of the Kolmogorov constant, consistent for three different methods of calculation, which agrees with the current estimates found in the numerical experiments [3,4].

The experimental setup we use has been described in previous papers [8–10] in the context of decaying turbulence. The flow is generated in a PVC cell, 15×15 cm. The cell is filled by two layers of NaCl solution, placed in a stable configuration, i.e., the heavier underlying the lighter. Permanent magnets are located just below the bottom of the cell. Their magnetization axis is vertical and they produce a magnetic field, 0.3 T in maximum amplitude, which decays over a typical length of 3 mm. An electric current is driven through the cell from one side to another. It has been shown [10] that the stratification considerably enhances the vertical momentum transfer so that, on the time scales relevant to a typical experiment, the flow measured at the free surface can be considered as two dimensional. The dissipation is provided by friction exerted by the bottom wall on the fluid, which can be parametrized by adding a linear term in the two-dimensional Navier-Stokes equations. This frictional term provides the infrared energy sink preventing the accumulation of energy in the largest scales of the flow. In the experiments presented here, the magnets are arranged such that the flow is initially concentrated around a prescribed wave number. We use a total fluid depth of 6 mm which corresponds to a friction strong enough to prevent the system from condensing in the lowest accessible modes. The imposed current, which, coupled to the magnetic field, defines the forcing, is a time series of impulses of constant amplitude and random sign. Each elementary impulse has a duration of 4 s, which is longer than the characteristic time of vertical transfers (which is shorter than 2 s; see [10]). The random-in-time forcing allows the imposition of an approximately zero net flow, which favors homogeneity. The current is switched on at

$t = 0$ and the flow is recorded for a typical time of 6 min: this corresponds to approximately 50 turnover times of the largest eddies. Once the flow has been recorded, the velocity fields are computed on 64×64 grids in the manner described in [8]. We typically compute 200 velocity fields for each experiment. Two-dimensional power spectra are obtained through fast Fourier transform and are further averaged along the angular coordinate to yield the radial spectra $E(k)$. The injection Reynolds number, defined using the root-mean-square velocity and the injection scale L_i, has a typical value of 100 in our experiments. All these characteristics are comparable in magnitude to the largest numerical studies performed on the subject.

A typical vorticity field, measured 2 min after the excitation is applied (i.e., well beyond the transient regime), is displayed in Fig. 1. The population of vortices includes a broad range of sizes; it would be hard to infer a single characteristic length from Fig. 1. This contrasts with freely decaying experiments where characteristic sizes can be defined at all times. Figure 1 also shows vorticity filaments which are produced by the breakup of sheared vortices submitted to the imposed random forcing. In this system, energy is transferred towards larger scales through the formation of clusters of like-sign vortices. In systems such as this, merging events are rare.

We now turn to the energy spectra. A typical series of spectra are shown in Fig. 2(a): the initial one ($\diamond$), computed 2 s after the current has been switched on, an intermediate one ($\circ$) averaged over a few fields covering a short time interval centered at $t = 10$ s, and the final spectrum ($\bullet$), averaged over 160 fields, well beyond the transient regime. The initial spectrum ($\diamond$) shows that the injection of energy is well localized in wave-number space. At later times, in the transient regime, the energy transfer from large to small wave numbers is signaled in Fourier space by the progressive building up of a spectrum with a $k^{-5/3}$ power law, as illustrated by the transient spectrum ($\circ$). The final spectrum ($\bullet$) displays a $k^{-5/3}$ behavior over a range slightly narrower than one decade (the black line corresponds to a calculated $-5/3$ scaling). In Fig. 2(b) the final spectrum of Fig. 2(a) is displayed compensated by the Kolmogorov scaling, $E(k)k^{5/3}$. Over the same range of scales as those for which the scaling law is observed in Fig. 2(a), a clear plateau is observed, which confirms that the spectral exponent is close to $-5/3$. For the present experiment, the most energetic wave number is $k_0 \approx 0.13$ cm^{-1}, which is larger than the wave number $2\pi/L$ based on the cell size. The stationarity of the system can be assessed from Fig. 3 which shows the temporal evolution of the kinetic energy E (black squares) and the enstrophy Z (empty squares). Time is rescaled by the mean turnover time of the large eddies t_R, which is typically 8 s. The figure shows that, except for the initial period of about 4 t_R, stationarity is well achieved in our experiments. Concerning statistical homogeneity and isotropy, things are less straightforward. Since the experiments are performed in a square box using a localized, poorly isotropic forcing, one expects the flow to be inhomogeneous and anisotropic at both ends of the spectrum, i.e., at the injection scale and on large scales. It is thus reasonable to bandpass filter the velocity field to reject the wave numbers outside the inertial range. After performing this filtering, we compute the angular energy spectrum and the field V' of the root-mean-square temporal fluctuations of the

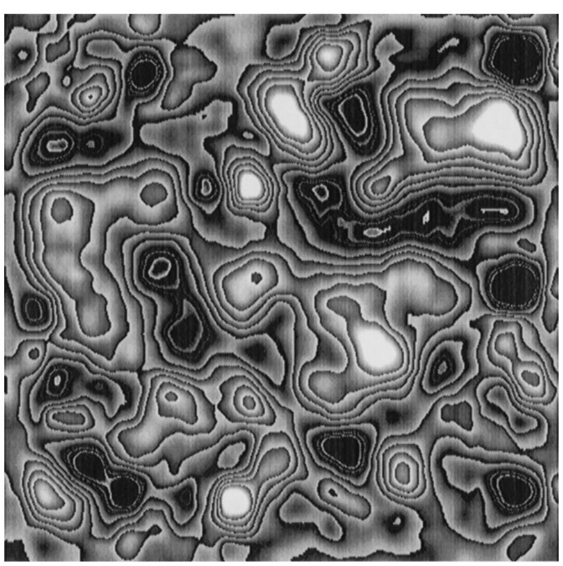

FIG. 1. Typical vorticity field during the stationary regime.

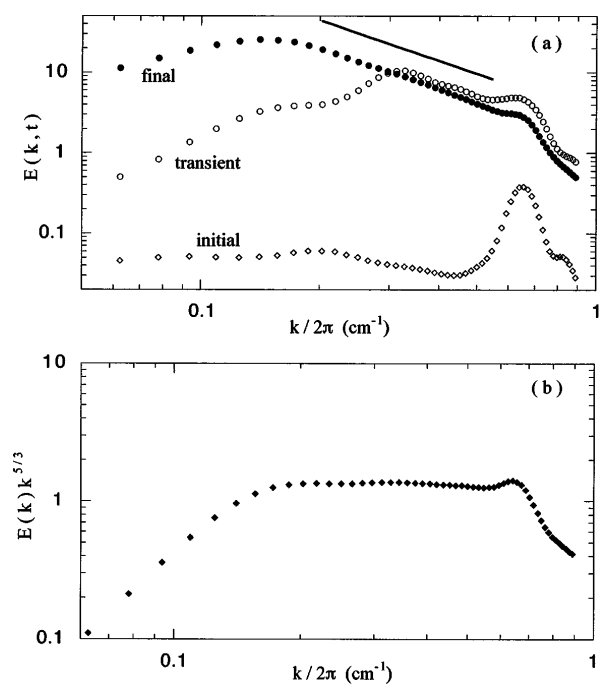

FIG. 2. Energy spectra: (a) temporal evolution and (b) compensated energy spectrum for the stationary regime.

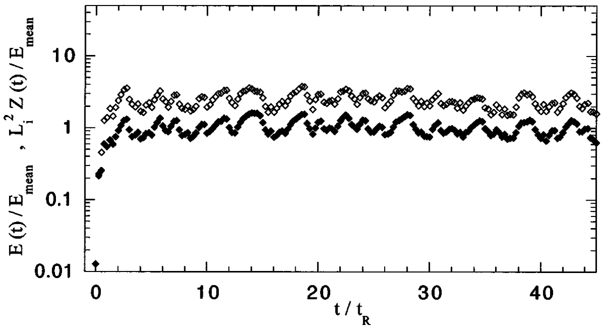

FIG. 3. Temporal evolutions of the total kinetic energy and enstrophy ($t_R = 8$ s; $L_i = 1.5$ cm).

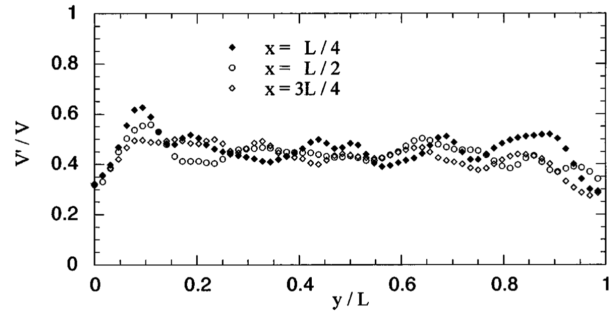

FIG. 5. Spatial repartition of the rms temporal fluctuations of the velocity.

velocities. The results are displayed in Figs. 4 and 5, respectively. The angular spectrum is rescaled by the total energy of the modes in the inertial range. The fluctuations, rescaled by the root-mean-square velocity, are drawn versus the y coordinate for three different values of the abscissa x. Coordinates are rescaled by the box size L. These figures show that both isotropy and homogeneity conditions are quite well satisfied in our experiments.

We now turn to the determination of the Kolmogorov constant. The expression of the energy spectrum reads [1]

$$E(k) = C_K \varepsilon^{2/3} k^{-5/3}, \tag{1}$$

where ε is the energy transfer rate and C_K is the two-dimensional Kolmogorov constant. Several methods can be proposed to determine the transfer rate from which the Kolmogorov constant is inferred. Determining ε from the measured overall dissipation leads to an overestimate of ε and consequently underestimates the Kolmogorov constant. The reason is that a sizable amount of the energy is drained out of the flow by bottom friction without taking part in the energy cascade. This might explain why, in [6], low values of C_K were found, at variance with the current numerical estimates. We thus use three different methods to determine ε. The first consists in computing the mean rate of transfer of kinetic energy $\Pi(k)$, and averaging it over the inertial range [see [1] for a detailed expression of $\Pi(k)$]. This method yields $C_K \approx 6.8$. The second method consists in tracking the evolution, in the transient period, of the most energetic wave number, i.e., the one corresponding to the maximum of the spectral energy, on plots such as those of Fig. 2. According to [1,4], if we assume that, during the transient phase, the energy spectrum has the form (1) in the range $k_0(t) \leq k \leq k_i$ and zero outside, we must have, on energy conservation grounds

$$k_0(t)^{-2/3} = k_i^{-2/3} + \frac{2\varepsilon^{1/3}}{3C_K} t, \tag{2}$$

where k_i is the injection wave number. We have determined the most energetic wave number k_0, averaged over six identical runs, as a function of time in the transient period. The result is shown in Fig. 6. Wave numbers are rescaled by the injection wave number and time is rescaled by the turnover time of the large eddies. The above equation (2) fits the experimental data well (see the black line in Fig. 6). The fit, together with the energy spectrum, yields $C_K \approx 5.8$. Finally, the third method consists in examining the linear growth of energy with slope ε in the transient period. For the same data, this gives $C_K \approx 6.3$. Summarizing these results, we propose the following estimate for the two-dimensional Kolmogorov constant:

$$C_K = 6.5 \pm 1 .$$

The uncertainty quoted here corresponds to the scattering of the values obtained with the alternative methods.

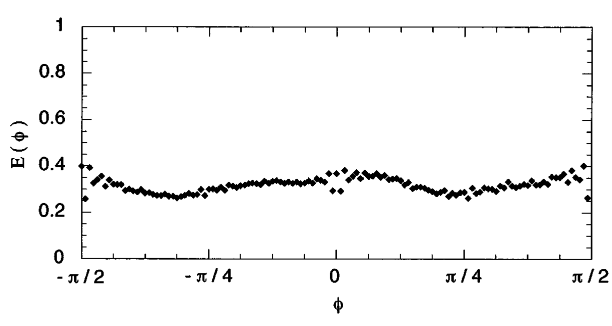

FIG. 4. Angular energy spectrum (inertial range).

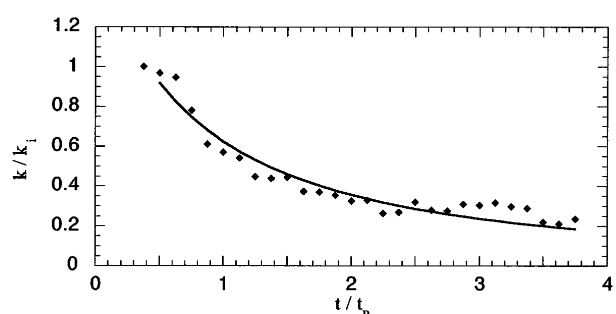

FIG. 6. Temporal evolution of the most energetic wave number ($t_R = 8$ s; $k_i/2\pi = 0.66$ cm^{-1}).

To conclude, the main result of this work is the first clear experimental observation of the inverse cascade of energy in a system which satisfies the usual assumptions of stationarity, homogeneity, and isotropy. The temporal evolution in the transient phase is in good agreement with the scaling proposed by Kraichnan [1]. We were able to obtain a two-dimensional Kolmogorov constant which is, within $\pm 15\%$, in agreement with the values currently found in high resolution numerical simulations. This may be a hint that it is universal, i.e., insensitive to the details of the forcing and the way energy is extracted at large scales, within some range of conditions. The cascade itself is found to be a stationary process over the time scales that we investigate. This does not preclude the possibility of existence of other regimes, at higher Reynolds numbers and lower friction, such as the so-called Bose-Einstein condensate, observed both numerically and experimentally [4,6,11].

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[11] We also observed the Bose-Einstein condensation in the present experiments using higher fluid depths, corresponding to lower friction.