# Transition to Traveling Waves from Standing Waves in a Rectangular Container Subjected to Horizontal Excitations

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We report an unexpected observation of a transition to traveling waves from standing waves. The waves are two dimensional and generated in a rectangular container excited by a horizontal sinusoidal motion along its length. The transition to traveling waves clearly indicates coupling between modes whose natural frequencies do not satisfy obvious resonance conditions. The observed phenomenon is beyond our current understanding of surface waves based on low-dimensional models and demands further study. It implies that care must be taken in developing approximate low-dimensional models for continuous systems in the presence of nonlinearity. [S0031-9007(97)03567-9]

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When a continuous system is subjected to periodic forcings, it is possible that the system response can be captured by just a few modes. This approximation has been widely used for systems which are not driven too far away from their equilibrium (hence the nonlinearity can be considered weak). If the frequency of the small amplitude forcing is close to the natural frequency of one normal mode, a single-degree-of-freedom model is often sufficient to capture the dynamics of the continuous system provided that the frequencies of other neighboring normal modes are noncommensurate to the forcing frequency.

When several natural frequencies are commensurate, energy transfer between these modes can occur. The dynamics of the interacting modes can nevertheless be captured by those of the coupled nonlinear oscillators containing the commensurate modes. Therefore, the natural frequencies of the normal modes determine the required degrees of freedom of the approximate low-dimensional models for the small amplitude forced responses.

The development of the above understanding has greatly simplified the analysis of surface waves in cylindrical containers. Models that include several resonant modes have satisfactorily predicted the dynamic behavior of waves under periodic excitations (Ciliberto and Gollub [1], Simonelli and Gollub [2], Feng and Sethna [3], Miles and Henderson [4]). It is in the context of these works that the results reported here are surprising and interesting.

Consider water waves in a rectangular container of length $L$ and width $W$. The depth of the water is $h$. The natural frequencies of the lowest modes are not significantly affected by the surface tension if the wavelength is large (on the order of 10 cm). In the absence of the surface tension, we have gravity waves whose dispersion relationship given in Whitham [5] is

$$\omega = (gk \tanh kh)^{1/2}. \tag{1}$$

For standing waves, the magnitude of the wave number vector is a function of the number of half waves (mode numbers), i.e., $k = \pi[(m/L)^2 + (n/W)^2]^{1/2}$. Table I shows the modal frequencies of the resulting "gravity waves," when the capillary effect is ignored, for $L = 22.86$ cm, $W = 12.7$ cm, and $h = 10.2$ cm 9 in. $\times$ 5 in. $\times$ 4 in.). The mode numbers $m$ and $n$ refer to the number of half waves along the directions of the length and the width, respectively. In particular, when either $m$ or $n$ is zero, the waves are two-dimensional waves. Although it is difficult to rule out the possibility that the ratios of some pair of modes are close to those of small integers, the frequencies of the two-dimensional waves seem to be noncommensurate with one another.

Since the natural frequencies of the two-dimensional modes are noncommensurate [although the natural frequencies of the third mode are nearly twice that of the first mode, they are not in resonance since both the natural frequencies and the wave number vectors must satisfy the resonance conditions; see O. M. Phillips, *The Dynamics of the Upper Ocean* (Cambridge University Press, Cambridge, 1967), p. 82], we expect that wave responses to small amplitude periodic forcing can be captured by the dynamics of a single-degree-of-freedom forced nonlinear oscillator. Instead we observed a transition phenomenon from two-dimensional one-mode standing waves to two-dimensional traveling waves which cannot yet be explained based on the low-dimensional models of surface waves.

Our experiment is motivated by the observation of large amplitude traveling waves in horizontally excited water tanks which are designed to dissipate vibrational energy resulting from wind-structure interactions in tall buildings and utility towers (Modi *et al.* [6]). Since the energy dissipations associated with the breaking of large amplitude traveling waves are the main interest of Modi *et al.* [6], the transition from the standing wave to the traveling wave, especially when the forcing amplitude is small and the weakly nonlinear approximations are valid, has not been examined. Yet this transition has important implications on using low-dimensional models to approximate the dynamics of continuous systems.

  

TABLE I. Natural frequencies of the lowest modes.

| Mode numbers | $m = 0$ | $m = 1$ | $m = 2$ | $m = 3$ | $m = 4$ | $m = 5$ |
|---|---|---|---|---|---|---|
| $n = 0$ | 0.000 | 1.738 | 2.604 | 3.200 | 3.696 | 4.132 |
| $n = 1$ | 2.463 | 2.643 | 3.030 | 3.456 | 3.870 | 4.260 |
| $n = 2$ | 3.506 | 3.572 | 3.750 | 4.000 | 4.287 | 4.587 |

Although observations exposing the inadequacies of the low-dimensional models have been reported before—Wu, Keolian, and Rudnick [7] have found solitons generated by periodic forcings—the transition to the traveling waves from the standing waves has not been reported. Furthermore, by the analogy drawn by Ockendon *et al.* [8] between surface waves and acoustic oscillations in closed resonators, the transition from standing waves to traveling waves may provide another explanation for the loss of stability in acoustic levitators (Rudnick and Barmatz [9]).

*Experimental configurations.*—Experiments are performed in a rectangular tank made of plexiglas which is 22.86 cm long, 12.7 cm wide, and 17.8 cm deep (9 in. $\times$ 5 in. $\times$ 7 in.). The tank is placed on top of a moving platform which is part of a hydraulic wave flume at the Ralph M. Parsons Laboratory of M.I.T. The length of the tank is aligned with the direction of motion of the platform. The platform is driven into sinusoidal oscillations in the horizontal plane by a sinusoidal signal from an HP 3325A synthesizer/function generator. The platform is mounted on tracks which are very stiff to keep the unwanted vibration in other directions to a negligible level.

Owing to the relatively large size of the tank, the surface tension effect is not important. However, the movement of the liquid contact line on the plexiglas walls shows stick-slip behavior. This causes the wave surface to appear rough. The stick-slip behavior is removed by adding 1 $cm^3$ of Kodak Photo-Flo 200 to the tap water in the tank.

The motion of the wave flume is controlled by the output of the function generator. The amplitude of the sinusoidal motion of the platform is proportional to the output voltage of the function generator. The proportionality coefficient depends on the frequency; the platform motion is bigger at low frequencies than at high frequencies for a fixed signal amplitude. This dependence on frequency is insignificant when we examine the transition of the wave type which occurs on a very narrow frequency interval (frequency change $< 0.03$ Hz).

Quantitative measurements are made using the following two methods: for steady periodic wave responses, the wave amplitudes are directly measured using a ruler; for unsteady wave responses, the wave amplitudes are measured by reading the wave height at a point 5 mm away from the side wall from the still picture frames recorded by a video camcorder. The measurements by either method are accurate up to $\pm 0.5$ mm.

*Transition among two dimensional waves.*—Table I shows the natural frequencies of a few low modes when the tank is filled with 10.2 cm (4 in.) of water. Under sinusoidal excitations along the length of the tank, only two-dimensional waves with mode numbers $m = 1$, $m = 3$, $m = 5$, etc. can be directly excited. Two-dimensional waves with even wave numbers are symmetric waves; they are not directly excited by the horizontal excitations which are antisymmetric.

In Fig. 1 we plot the wave amplitude responses when the signal from the HP function generator is held fixed at 90 mV (peak-to-peak) and the forcing frequencies are slowly changed from 0.5 to 4.05 Hz. The wave amplitude responses show typical resonance peaks near the natural frequencies of the first mode and the third mode. Near these resonance frequencies, the wave responses are dominated by a single mode. The other modes are too small to be noticeable. The wave responses of the first mode and the third mode are standing waves characterized by one and three clearly observable nodes, respectively. These nodes are fixed relative to the water tank. At the frequencies at which the wave responses change from the first mode to the third mode, the wave amplitude is very small. In other words, the resonances of the first mode and the third mode are well separated. Near each resonance frequency, the wave response is similar to that of a single-degree-of-freedom nonlinear oscillator. The nonlinearity is manifested by the slight asymmetry of the response curve, which is more easily observable in Fig. 2.

In Fig. 2 we plot more detailed wave responses for forcing frequencies between 3.05 and 3.29 Hz. This frequency interval corresponds to the interval on which mode 3 standing wave responses have been observed in Fig. 1. The solid dots with an interpolating line passing through them are the amplitudes of the third mode standing waves when the forcing amplitude is 55 mV. The peak-to-peak amplitude of the moving

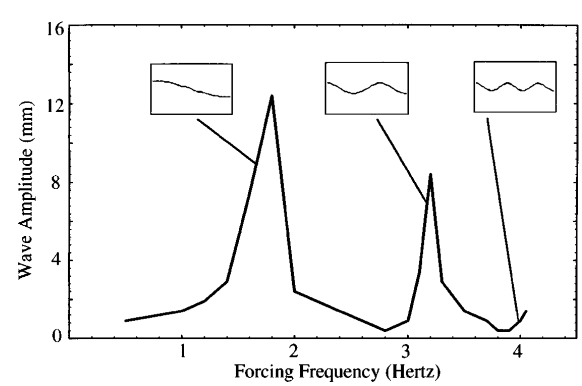


FIG. 1. Wave responses at different driving frequencies on the interval of 0.5 Hz and 4.05 Hz.

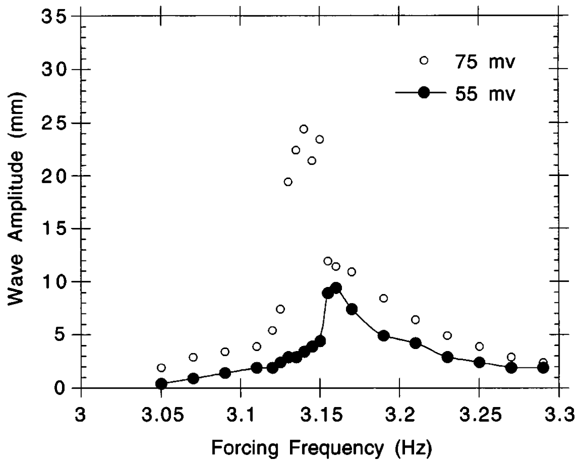


FIG. 2. Wave amplitudes when the forcing amplitudes are held fixed at 55 and 75 mV, respectively. The five empty circles represent the maximum wave heights of nonsteady waves measured during a 1 min time interval.

platform corresponding to this signal is about 0.3 mm. The resonant peak occurs near 3.16 Hz as compared to the theoretical value of 3.20 Hz. Moreover, the interpolating line leans to the left, indicative of a softening nonlinearity (Ockendon *et al.* [8]).

The amplitudes of wave responses when the forcing signal is 75 mV are shown as open circles in Fig. 2. Exploration of the wave responses on a finer frequency interval reveals that there is a very narrow frequency interval, 3.13 to 3.15 Hz, on which the wave response is nonstationary. A typical nonsteady wave response is shown in Fig. 3. Here the unsteady wave was recorded by a video camcorder after the forcing amplitude and frequency have been held fixed for several minutes. For unsteady waves, the wave amplitudes are not defined; hence the five open circles in Fig. 2 correspond to the largest wave height measured during a time interval about 1 min long.

As the wave responses change from steady third-mode standing waves to nonsteady waves, the wave type also changes. The nonsteady waves are not standing waves of the third mode with modulating amplitudes, which could have been easily identified by the three nodes that are fixed relative to the tank; instead, they are traveling waves with a peak moving from one side of the tank to the other. Four still frames recorded by a camcorder as the wave peak moves from right to left are shown in Fig. 4. The time interval between successive frames is about 0.3 s. The traveling waves are not solitary waves. The period of one round trip of the traveling wave is much longer than the period of the forcing: at forcing frequency 3.14 Hz, the traveling wave makes 33 to 34 round trips in 1 min. This corresponds to a wave speed about 0.26 m/s, which is close to the group speed of the gravity wave $(0.25\ \mathrm{m/s})$ for $k = 2\pi/\lambda$ where $\lambda$ is the wavelength of the third mode.

The transition from steady standing waves to nonsteady traveling waves is an easily reproducible phenomenon. The experiments have been repeated several times. During each run, the water tank is mounted on and carefully aligned with the platform of the wave flume. Fresh tap water is filled to a desired height and about 1 cm$^3$ of Kodak Photo-Flo 200 solution is added to the water. At the end of each experiment, the water is emptied from the tank. The tank and the function generator are stored away until the next experiment, which usually takes place several days later.

Figure 5 shows the partition of the parameter space of the forcing amplitude and frequency. The solid dots denote traveling wave responses. Data for this figure are obtained by fixing the forcing amplitude and changing the forcing frequencies at 0.005 Hz increments. We usually wait several minutes before concluding that the wave is one type or the other. Occasionally, we wait more than 30 min to rule out the possibility that the nonsteady traveling waves are just some very long transient responses. We observe that as the forcing amplitude increases, the frequency interval on which traveling waves

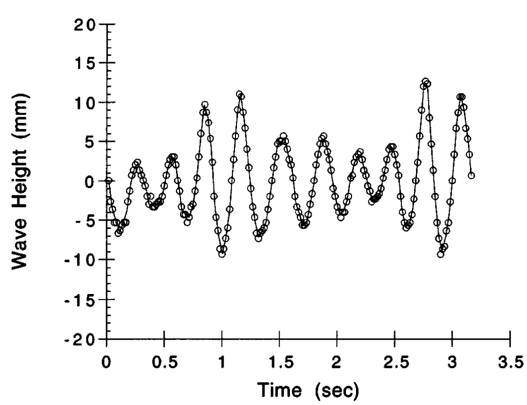


FIG. 3. The nonsteady wave height as a function of time observed from consecutive still video frames.

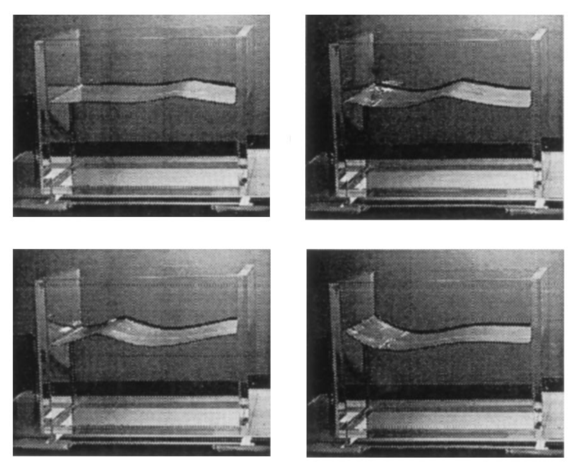

FIG. 4. Four still video frames showing the movement of the peak of the traveling waves. The peak travels from right side wall to the left side wall. The time sequences are (i) top-left, (ii) top-right, (iii) bottom-left, and (iv) bottom-right. The time interval between the two consecutive frames is about 0.3 s.

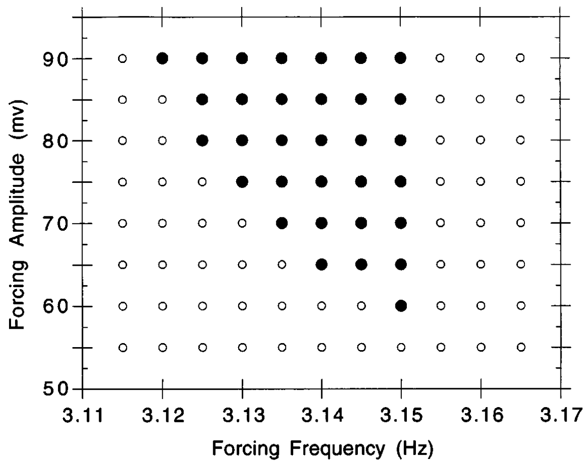


FIG. 5. Types of wave responses for different combinations of the forcing amplitudes and frequencies. Empty circles denote parameters for which steady standing waves are observed, and solid circles denote parameters for which traveling waves are observed.

are observed increases. We explored forcing amplitudes up to 90 mV. The corresponding peak-to-peak displacement of the water tank is about 0.5 mm. Responses to even larger forcing amplitudes are not explored since we are interested in small amplitude wave responses (wave amplitude less than 20 mm in a tank of 228.6 mm long) for which weakly nonlinear models are expected to give good approximations. For these forcing amplitudes, the hysteresis of the transition from one type to the other as the frequency is changed is too small (less than 0.005 Hz) to be easily quantified.

The transitions to traveling waves have also been observed when the water depth is 7.62 cm (3 in.) and 12.7 cm (5 in.) respectively. We are not able to observe the traveling waves when the water depth is 5.08 cm (2 in.) since our attention is focused on the forcing frequencies which are close to the resonance frequency of the third mode. It has been brought to our attention that, for shallow water, when forced at a much lower frequency, Bridges [10] has observed a traveling hydraulic jump resulting from cnoidal standing waves. Since the resonance frequencies of the high modes are integer multiples of the fundamental mode for large amplitude cnoidal standing waves in shallow water, complicated modal coupling is thus not surprising.

Low-dimensional models have been successfully employed to study the dynamic responses of surface waves in finite containers (Ciliberto and Gollub [1], Feng and Sethna [3], Miles and Henderson [4]). These low-dimensional models assume from the outset that the free surface is a superposition of standing waves of several resonant modes. These selective modes act like coupled nonlinear oscillators. Their responses to the periodic forcing capture the overall system responses. Among infinitely many normal modes, only modes which are in resonance with the external forcing and with one another are known to have an effect on the system dynamics. This drastically simplifies the analysis of weakly nonlinear systems.

Throughout these experiments, the wave responses remain two-dimensional even when the waves become nonsteady traveling waves. The geometry corresponding to our experiments is such that none of the two-dimensional modes are resonantly coupled to the third two-dimensional mode. The occurrence of the transition to traveling waves points to new possible solutions in this simple well-studied problem. Although other unexpected solutions have been found in this context (Wu, Keolian, and Rudnick [7] report the generation of nonpropagating solitons in similar wave experiments), the transition to the traveling waves from standing waves has not been reported before.

The observed transition may also suggest instabilities of standing waves in other physical context. Ockendon *et al.* [8] draw an analogy between the sloshing of liquids in a horizontally oscillated, rectangular tank and one-dimensional radially symmetric acoustic oscillations in a closed resonator. This analogy suggests that the standing waves in an acoustic levitator may become unstable. The loss of stability is likely to cause the loss of the ability to levitate objects. This can provide another explanation to the onset of translational instability in single-mode acoustic levitators (Rudnick and Barmatz [9]).

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