Higher-Order Corrections to Sirlin's Theorem in $\mathcal{O}(p^6)$ Chiral Perturbation Theory

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We present the results of the first two-loop calculation of a form factor in full SU(3) × SU(3) chiral perturbation theory. We choose a specific linear combination of π^+, K^+, K^0, and $K\pi$ form factors (the one appearing in Sirlin's theorem) which does not get contributions from order p^6 operators with unknown constants. For the charge radii, the corrections to the previous one-loop result turn out to be significant. To clearly identify the two-loop effects, more accurate measurements of the kaon and pion electromagnetic charge radii would be desirable. [S0031-9007(97)04524-9]

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Chiral perturbation theory (ChPT) has been applied with great success to low energy hadronic phenomena [1–3]. Presently there is an emerging effort to extend the calculations to two-loop order so as to allow quantitative comparison with experiments and test the convergence properties of ChPT. Until now most of the two-loop results have been obtained in the chiral SU(2) × SU(2) limit which is obviously a serious limitation as K mesons are excluded from loops. To our knowledge a complete SU(3) × SU(3) calculation exists so far only for two-point functions of current correlators [4]. In this Letter we will present the results of the first full SU(3) × SU(3) form factor calculation. We choose a specific combination of weak and electromagnetic meson form factors which does not involve arbitrary renormalization constants of new operators.

ChPT is formulated in terms of an effective Lagrangian involving an increasing number of covariant derivatives, external fields (including quark mass terms), and field strength tensors,

$$\mathcal{L}_{\text{eff}} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \dots . \tag{1}$$

The lowest-order term is [1]

$$\mathcal{L}^{(2)} = \frac{F^2}{4}\,\text{Tr}(D_\mu U D^\mu U^\dagger) + \frac{F^2}{4}\,\text{Tr}(\chi U^\dagger + U\chi^\dagger), \tag{2}$$

with $U(x) = \exp[i\Phi(x)/F]$, where Φ is the 3×3 matrix made up of the Goldstone fields (π, K, η), F is the pion decay constant in the chiral limit, and χ is related to the quark mass matrix (for details see [2]). The term $\mathcal{L}^{(4)}$ is of order p^4 and involves 10 new operators which are to be renormalized by imposing the same number of independent experimental input data. The renormalization constants L_1 to L_{10} are commonly defined in dimensional regularization. The operators of $\mathcal{L}^{(6)}$ have been exhaustively analyzed in [5] and found to be 143 in number. The corresponding number of free constants seems to be prohibitive. There are, however, subsets of experiments, such as the weak and electromagnetic form factors of mesons, which involve only a small number of new renormalization constants. ChPT to order p^6 then leads to relations between and predictions of specifics of their t dependence. In one combination of these form factors the $\mathcal{L}^{(6)}$ constants all cancel. This is the combination entering Sirlin's relation [6] and which should vanish in the chiral limit.

The relevant vector current form factors are defined as follows:

$$\langle \pi^+, p'|J_\mu|\pi^+, p\rangle = (p + p')_\mu F^\pi(t), \tag{3}$$

$$\langle K, p'|J_\mu|K, p\rangle = (p + p')_\mu F^K(t), \tag{4}$$

$$\langle \pi^-, p'|\bar{u}\gamma_\mu s|K^0, p\rangle = (p + p')_\mu f_+^{K\pi}(t) + (p - p')_\mu f_-^{K\pi}(t), \tag{5}$$

where $t = (p' - p)^2$ and J_μ is the electromagnetic current carried by the light quarks, $J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s$. Sirlin's low-energy theorem then states that, up to second order in the quark mass difference $m_s - \hat{m}, \hat{m} = \frac{1}{2}(m_u + m_d)$, the linear combination

$$\Delta(t) = \frac{1}{2}\,F^{\pi^+}(t) + \frac{1}{2}\,F^{K^+}(t) + F^{K^0}(t) - f_+^{K\pi}(t) \tag{6}$$

vanishes. The effect of heavy quarks in the electromagnetic current is neglected. Sirlin's relation generalizes the Ademollo-Gatto theorem [7] to $t \neq 0$. For $t = 0$, Eq. (6) yields no prediction as $f_+^{K\pi}(0)$ still depends on unknown constants of $\mathcal{L}^{(6)}$. In the relation for the charge radii (and higher Taylor coefficients), however, all arbitrary constants cancel and an unambiguous prediction remains.

The diagrams contributing to the t dependence of the form factors at order p^6 are represented symbolically in Fig. 1. There are further graphs which do not depend on t and are omitted here. The $\mathcal{O}(p^6)$ contributions arise from the lowest-order diagrams 1(a) and 1(b) owing to wave function, mass, and decay constant renormalization,

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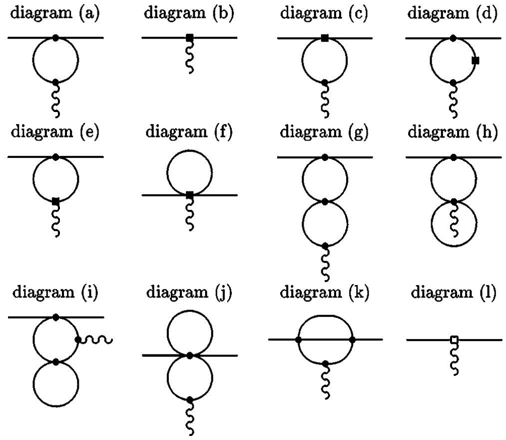

FIG. 1. The form factor diagrams with t dependence. $\mathcal{L}^{(2)}$ vertices are denoted by filled circles (●), $\mathcal{L}^{(4)}$ vertices by filled squares (■), and an $\mathcal{L}^{(6)}$ vertex by an open square (□).

from the one-loop diagrams 1(c)–1(f) with one vertex from $\mathcal{L}^{(4)}$, from the reducible two-loop diagrams 1(g)–1(j), from the irreducible two-loop diagram [1(k)], and from the tree graph 1(l) with one vertex from $\mathcal{L}^{(6)}$, which yields no contributions to Sirlin's linear combination $\Delta(t)$. Diagrams 1(b), 1(f), and 1(l) are polynomial in t due to the derivative couplings in the vertices.

All reducible diagrams 1(a)–1(j) involve only well-known one-loop integrals [8] (calculated to order $\varepsilon = 2 - D/2$), i.e., the massive one-loop tadpole

$$A(m^2) = \mu^{4-D} \int \frac{d^D k}{i(2\pi)^D} \frac{1}{k^2 - m^2}, \tag{7}$$

and the one-loop two-point functions $B_{21}(q^2; m_1^2, m_2^2)$ and $B_{31}(q^2; m_1^2; m_2^2)$ defined by the tensor decompositions

$$\mu^{4-D} \int \frac{d^D k}{i(2\pi)^D} \frac{k^\mu k^\nu}{[(k+q)^2 - m_1^2][k^2 - m_2^2]} =: q^\mu q^\nu B_{20}(q^2; m_1^2, m_2^2) + g^{\mu\nu} B_{21}(q^2; m_1^2, m_2^2), \tag{8}$$

$$\mu^{4-D} \int \frac{d^D k}{i(2\pi)^D} \frac{k^\mu k^\nu k^\rho}{[(k+q)^2 - m_1^2][k^2 - m_2^2]} =: q^\mu q^\nu q^\rho B_{30} + (g^{\mu\nu} q^\rho + g^{\mu\rho} q^\nu + g^{\nu\rho} q^\mu) B_{31}. \tag{9}$$

In terms of these integrals Sirlin's relation reads

$$\Delta(t) = \delta_4(t) + \delta_6(t) + \ldots, \tag{10}$$

where δ_4 is the $\mathcal{O}(p^4)$ result [3],

$$\delta_4(t) = \frac{1}{8F^2}\{-3A(m_\eta^2) + 4A(m_K^2) - A(m_\pi^2) + 12B_{21}(q^2; m_\eta^2, m_K^2) - 20B_{21}(q^2; m_K^2, m_K^2) + 12B_{21}(q^2; m_K^2, m_\pi^2) - 4B_{21}(q^2; m_\pi^2, m_\pi^2)\}, \tag{11}$$

and δ_6 is the $\mathcal{O}(p^6)$ contribution. Up to a constant which is irrelevant for the t dependence of $\Delta(t)$, δ_6 is given as a sum

$$\delta_6(t) = \mathrm{Red}_1(t) + \mathrm{Red}_2(t) + \mathrm{Irr}(t) + \mathrm{const}, \tag{12}$$

where the reducible one-loop part $\mathrm{Red}_1(t)$ collects all terms involving $\mathcal{L}^{(4)}$ parameters $L_1, \ldots, L_{10}$,

$$\begin{aligned}\mathrm{Red}_1(t) = \frac{1}{4F^4}\{&-3A(m_\eta^2)q^2L_9 + 4A(m_K^2)q^2L_9 - A(m_\pi^2)q^2L_9 + 16B_{31}(q^2; m_\eta^2; m_K^2)L_3(m_K^2 - m_\pi^2)\\ &- 16B_{31}(q^2; m_K^2; m_\pi^2)(8L_1 + 4L_2 + L_3)(m_K^2 - m_\pi^2)\\ &+ 4B_{21}(q^2; m_\eta^2, m_K^2)[2L_3(m_K^2 - m_\pi^2 - 3q^2) + 12L_5 m_\pi^2 + 3q^2L_9]\\ &+ 4B_{21}(q^2; m_K^2, m_K^2)[8q^2L_1 - 4q^2L_2 + 10q^2L_3 - 16L_4m_K^2 - 20L_5m_\pi^2 - 5q^2L_9]\\ &+ 4B_{21}(q^2; m_\pi^2, m_\pi^2)[8q^2L_1 - 4q^2L_2 + 2q^2L_3 - 16L_4m_\pi^2 - 4L_5m_\pi^2 - q^2L_9]\\ &+ 4B_{21}(q^2; m_K^2, m_\pi^2)[-16L_1(m_K^2 - m_\pi^2 + q^2) - 8L_2(m_K^2 - m_\pi^2 - q^2)\\ &- 2L_3(m_K^2 - m_\pi^2 + 3q^2) + 16L_4(m_K^2 + m_\pi^2) + 12L_5m_\pi^2 + 3q^2L_9]\},\end{aligned} \tag{13}$$

$\mathrm{Red}_2(t)$ denotes the reducible two-loop parts,

$$\begin{aligned}\mathrm{Red}_2(t) = \frac{1}{144F^4}\{&99A(m_\eta^2)B_{21}(q^2; m_\eta^2, m_K^2) + 102A(m_\eta^2)B_{21}(q^2; m_K^2, m_K^2)\\ &+ 75A(m_\eta^2)B_{21}(q^2; m_K^2, m_\pi^2) - 6A(m_\eta^2)B_{21}(q^2; m_\pi^2, m_\pi^2) + 318A(m_K^2)B_{21}(q^2; m_\eta^2, m_K^2)\\ &- 924A(m_K^2)B_{21}(q^2; m_K^2, m_K^2) + 350A(m_K^2)B_{21}(q^2; m_K^2, m_\pi^2) - 104A(m_K^2)B_{21}(q^2; m_\pi^2, m_\pi^2)\\ &+ 483A(m_\pi^2)B_{21}(q^2; m_\eta^2, m_K^2) - 678A(m_\pi^2)B_{21}(q^2; m_K^2, m_K^2) + 475A(m_\pi^2)B_{21}(q^2; m_K^2, m_\pi^2)\\ &- 190A(m_\pi^2)B_{21}(q^2; m_\pi^2, m_\pi^2) - 324B_{21}(q^2; m_\eta^2, m_K^2)^2 + 1008B_{21}(q^2; m_K^2, m_K^2)^2\\ &+ 144B_{21}(q^2; m_\pi^2, m_\pi^2)^2 - 648B_{21}(q^2; m_\eta^2, m_K^2)B_{21}(q^2; m_K^2, m_\pi^2) - 324B_{21}(q^2; m_K^2, m_\pi^2)^2\\ &+ 144B_{21}(q^2; m_K^2, m_K^2)B_{21}(q^2; m_\pi^2, m_\pi^2)\},\end{aligned} \tag{14}$$

and Irr(t) is the irreducible two-loop contribution from diagram 1(k). Diagram 1(k) cannot be calculated analytically, unless all masses are equal. Instead, for arbitrary masses and tensor numerators, it can be reduced via dispersion techniques to a one-dimensional integral which is done numerically [9,10]. Its contribution to the charge radius

$$\langle r^2\rangle_{\text{Sirlin}} := 6\left.\frac{d\Delta}{dt}\right|_{t=0} \tag{15}$$

turns out to be small if one uses the generalized Gasser-Leutwyler renormalization scheme, i.e., multiplication of each $\mathcal{O}(p^6)$ contribution with the $\overline{\text{MS}}$-type factor

$$[\varepsilon(4\pi)^\varepsilon\Gamma(-1+\varepsilon)]^2 = \exp\left[2\varepsilon(\gamma - 1 - \log 4\pi) - \varepsilon^2\left(\frac{\pi^2}{6}+1\right) + \mathcal{O}(\varepsilon^3)\right]. \tag{16}$$

For $\mu = 770$ MeV, $F_\pi = 92.4$ MeV, $m_K = 495$ MeV, $m_\pi = 135$ MeV, and $m_\eta = 548.8$ MeV, we find

$$\langle r^2\rangle_{\text{Sirlin}} = [0.006\ \text{fm}^2]^{\mathcal{O}(p^4)} + [0.017(3)\ \text{fm}^2]^{\text{red}\,\mathcal{O}(p^6)} + [-0.002\ \text{fm}^2]^{\text{irr}\,\mathcal{O}(p^6)} \tag{17}$$

$$= (0.021 \pm 0.003)\ \text{fm}^2, \tag{18}$$

where the error is due to uncertainties in the $\mathcal{L}^{(4)}$ parameters L_i involved. In the $\mathcal{O}(p^4)$ result the parameter F and the meson masses occurring in δ_4, cf. Eq. (11), are taken as their physical values; $\mathcal{O}(p^6)$ renormalization effects are lumped into δ_6 and are small because δ_4 is small.

This is to be compared with the experimental point $\langle r^2\rangle_{\text{exp}} = -(0.025 \pm 0.041)$ fm^2 which is based on the data [11–14]

$$\langle r^2\rangle^{\pi^+} = (0.439 \pm 0.008)\ \text{fm}^2, \tag{19}$$

$$\langle r^2\rangle^{K^+} = (0.34 \pm 0.05)\ \text{fm}^2, \tag{20}$$

$$\langle r^2\rangle^{K^0} = -(0.054 \pm 0.026)\ \text{fm}^2, \tag{21}$$

$$\langle r^2\rangle_+^{K\pi} = (0.36 \pm 0.02)\ \text{fm}^2. \tag{22}$$

It is seen from Eq. (17) that the $\mathcal{O}(p^6)$ corrections to Sirlin's theorem are larger than the $\mathcal{O}(p^4)$ ones. On the other hand, the very fact that the $\mathcal{O}(p^6)$ counterterms cancel in Sirlin's relation may render the two-loop result (and *a fortiori* the old one-loop result [3]) unreliable. This is because the unknown $\mathcal{O}(p^8)$ counterterms might contain the important resonance physics effects which have been found to dominate counterterms at lower chiral order. The question of convergence of chiral perturbation theory in Sirlin's relation can be answered with more confidence if agreement between our result and experiment is found on the basis of more accurate data.

We have actually calculated the complete t dependence of $\Delta(t)$, but we find only slight deviations from linearity [10]. From Fig. 2 we see that Sirlin's relation and the $\mathcal{O}(p^4)$ prediction are consistent, within one standard deviation, with current experiments. The $\mathcal{O}(p^6)$ result tends to increase the difference and shows a deviation of 1.1σ.

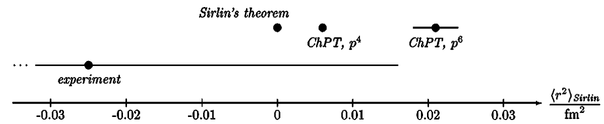

FIG. 2. The charge radius of Sirlin's linear combination, $\langle r^2\rangle_{\text{Sirlin}} = \frac{1}{2}\langle r^2\rangle^{\pi^+} + \frac{1}{2}\langle r^2\rangle^{K^+} + \langle r^2\rangle^{K^0} - \langle r^2\rangle_+^{K\pi}$: Sirlin's theorem, the $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ predictions of ChPT, and the experimental value.

In order to verify accurately these predictions and to test the higher-order contributions, more precise experiments are required. The main experimental uncertainties lie in the kaon charge radii and the slope of the $K_{\ell 3}$ form factor, which ought to be remeasured with higher accuracy. In particular, it may be argued [3] that the K^+ charge radius $\langle r^2\rangle^{K^+}$ should be larger than $\langle r^2\rangle_+^{K\pi}$ which would bring prediction and experiment into better agreement.

We have checked our calculations in several ways:

(i) In the special case of all masses equal, the irreducible two-loop integrals were compared to known analytic results [15].

(ii) The electromagnetic form factors have to satisfy the Ward identity. This holds separately for the group of reducible and the group of irreducible diagrams.

(iii) Nonpolynomial divergences have to disappear in the sum of all loop diagrams.

In this Letter we have reported the results of the first two-loop or $\mathcal{O}(p^6)$ calculation of a form factor in full chiral SU(3) × SU(3) perturbation theory. We chose a particular combination of weak and electromagnetic form factors due to Sirlin which is independent of the new arbitrary renormalization constants of $\mathcal{L}^{(6)}$ (except at $t = 0$). The correction to the previous one-loop result [3] turns out to be significant. Comparison of Sirlin's linear combination of charge radii with data is inconclusive due to large experimental uncertainties. An accurate comparison of Sirlin's relation with the data requires a more precise measurement of the kaon charge radii and the slope of the $K_{\ell 3}$ form factor. A significant improvement in the precision of the charged kaon form factor should be feasible in the future COMPASS experiment [16].

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