Critical Behavior of Random-Bond Potts Models

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The effect of quenched impurities on systems undergoing first-order phase transitions is studied within the framework of the q-state Potts model. For large q a mapping to the random-field Ising model explains the absence of any latent heat in 2D, and suggests that for $d > 2$ such systems exhibit a tricritical point with an exponent ν related to those of the random-field model by $\nu = \nu_{\mathrm{RF}}/(2 - \alpha_{\mathrm{RF}} - \beta_{\mathrm{RF}})$. In 2D we analyze the model using finite-size scaling and conformal invariance, and find a continuous transition with a ratio β/ν which varies continuously with q, and a weakly varying exponent $\nu \approx 1$. We find strong evidence for the multiscaling of the correlation functions. [S0031-9007(97)04550-X]

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Although the effect on the critical behavior of adding quenched bond randomness to classical systems whose pure version undergoes a continuous phase transition is well understood in terms of the Harris criterion [1], the analogous situation when the pure transition is first order is less well studied. Following earlier work of Imry and Wortis [2], Aizenman and Wehr [3] and Hui and Berker [4] argued that in 2D any amount of randomness should lead to a vanishing of the latent heat. The arguments leading to this conclusion are analogous to those used by Imry and Ma [5] for the absence of any spontaneous magnetization in the random *field* Ising model (RFIM) for $d = 2$: the bond randomness couples to the local energy density, which is different for the coexisting phases of the pure model, in the same manner that the random field couples to the local magnetization of the RFIM. The vanishing of the latent heat should be accompanied by a divergent correlation length, and, if so, the question arises as to which universality class(es) the corresponding continuous transition belongs. A suitable model in which to study this is the q-state Potts model, whose pure version in 2D undergoes a first-order transition for $q > 4$, otherwise being continuous. Chen, Ferrenberg, and Landau [6] undertook an extensive Monte Carlo investigation of the case $q = 8$. In addition to confirming the continuous nature of the transition, these authors extracted numerical values of the critical exponents which appear to be consistent with those of the *pure* 2D Ising model. Similar values have also been claimed for the case $q = 4$, when the pure transition is continuous [7]. This disagrees with the predictions of Ludwig and Cardy [8], Ludwig [9], and Dotsenko *et al.* [10], who find a new random fixed point for $q > 2$, based on an expansion in powers of $q - 2$.

As will become clear, many of our results generalize, but let us for definiteness consider a Potts model on the square lattice with degrees of freedom s_i taking q values, and a reduced Hamiltonian $-\sum_{ij} K_{ij}\delta_{s_i s_j}$, where the sum is over nearest neighbor pairs. The ferromagnetic couplings K_{ij} are quenched random variables, taking the values K_1 and K_2, each with probability $\frac{1}{2}$. When $(e^{K_1} - 1)(e^{K_2} - 1) = q$ this model is, on average, self-dual, and, if the transition is unique, is therefore at its critical point [11]. It is useful to parametrize $e^{K_{ij}} - 1 = u_{ij} = q^{(1/2)+w_{ij}}$, where $w_{ij} = \pm w$, and $w > 0$ measures the strength of the randomness. The partition function of this model may be mapped onto that of the random cluster model [12], in which each bond of the lattice is either occupied, when it is counted with weight u_{ij}, or empty, in which case it is counted with weight 1. The partition sum is over all such configurations, in which each connected cluster of sites is weighted by a factor q. Let us first consider the pure model, with $w = 0$. In the limit $q \to \infty$, the sum over configurations is dominated by only two: the *empty* lattice, in which no bonds are occupied, which contributes a factor q^N, where N is the total number of sites, and the *full* lattice, with a weight $(\sqrt{q})^{2N}$, since the number of bonds is $O(2N)$. All other configurations are down by powers of $q^{1/2}$. At the self-dual point, there are therefore two coexisting states with identical bulk free energy and different internal energy densities, indicating, as expected, that the transition has a nonvanishing reduced latent heat per bond $\sim\frac{1}{2}\ln q$. For the pure model, this analysis may be extended to take into account higher order corrections in $q^{-1/2}$, with no essential change in the physical picture. Now consider an *interface* between these two phases. For large q, the lowest energy interface is parallel to a lattice direction, say the x axis, and is such that all the bonds with $y \le$ some integer are occupied, and those above this are empty (or vice versa). There will also be entropic fluctuations $y = h(x)$ of this interface, described by the usual solid-on-solid interfacial Hamiltonian, proportional to the length of the interface. The interfacial tension for large q is $\sigma \sim \frac{1}{4}\ln q$, independent of the local shape of the interface. This is to be compared with $\sigma \sim 2J$ between the *ordered* phases of a low temperature Ising model with reduced exchange coupling J.

Now consider the effect of adding bond randomness to the random cluster model. Each configuration of the interface will be weighted by an energy $\sum_x \sum_{y<h(x)} \times w(x,y) \ln q$, where (x,y) labels bond positions. This may be rewritten, up to a term independent of $h(x)$, as $\frac{1}{2}\sum_x (\sum_{y<h(x)} - \sum_{y>h(x)}) w(x,y) \ln q$. Compared with the energy of an RFIM interface with spins $s(x,y) = \pm 1$ coupled to a reduced random field $h(x,y) = \pm h_{\rm RF}$, the interfacial models are identical with the correspondence $J \leftrightarrow \frac{1}{8}\ln q$, and $h_{\rm RF} \leftrightarrow \frac{1}{2} w \ln q$. In addition, the imposition of a uniform reduced magnetic field h on the RFIM, which distinguishes between the two coexisting phases, is seen to be equivalent to a deviation $t \equiv (T - T_c)/T_c$ in the *temperature* variable away from the critical self-dual point. Since this couples to the energy density we find the correspondence $h \leftrightarrow \frac{1}{4} t \ln q$.

Of course, this is strictly valid only as $q \to \infty$. At finite q the q dependence of cluster configurations with more complicated topologies is not simply accounted for by the interfacial tension. For the same reason, the mapping is not between *bulk* configurations of the two models. However, it will be argued that certain universal properties are controlled by an a renormalization group (RG) fixed point at infinite q, and for these the mapping should be asymptotically exact. Although this has been described in terms of a 2D self-dual model, it should be clear that it is more general: lack of self-duality corresponds to a skewness in the distribution of the random fields $h(x,y)$, which may be compensated by adding a suitable uniform field (corresponding to a shift in the T_c of the Potts model), and, similarly, higher dimensions may be taken into account by appropriately replacing $\sqrt{q}$ by $q^{1/d}$.

The RG properties of the interface in the RFIM near $d = 2$ have been well studied [13]. When translated into the variables of the random-bond Potts model, the flow equations have the form

$$dw/dl = -(d/2 - 1)w + Aw^3 + \ldots, \quad (1)$$

$$d(\ln q)^{-1}/dl = -(\ln q)^{-1}[(d-1) - Aw^2 + \ldots], \quad (2)$$

$$dt/dl = t(1 + Aw^2 + \ldots), \quad (3)$$

where $A > 0$ is a nonuniversal constant. Corrections to these equations are supposed to be higher order in w and in $q^{-1/2}$. The RG flows for $d > 2$ and the consequent phase diagram are shown in Fig. 1. In the pure models, for $q >$ some $q_2(d)$ (low T in the RFIM), there is phase coexistence with a nonvanishing latent heat (spontaneous magnetization), controlled by a fixed point at infinite q ($T = 0$). For $d > 2$ this persists into the shaded region, bounded by a line of tricritical points where the latent heat vanishes. The universal behavior along this line is controlled by the fixed point R at $w = O((d-2)^{1/2})$ and infinite q. Using the correspondence $t \leftrightarrow \frac{1}{2} h \cdot T_{\rm RF}$, we conclude that the thermal eigenvalue ν^{-1} of the random-bond problem is related to the RG eigenvalues and exponents of the RFIM by the scaling relations $\nu^{-1} = y_h - \theta = (2 - \alpha_{\rm RF} - \beta_{\rm RF})/\nu_{\rm RF}$, where $-\theta$ is the eigenvalue

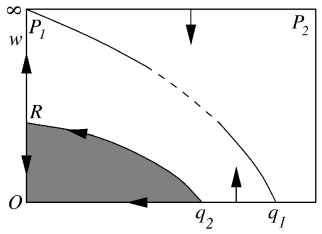

FIG. 1. Schematic phase diagram in the critical surface for $d > 2$. q increases to the left, and w is the disorder strength, with P_1P_2 being the percolation limit. RG flows are indicated. The latent heat is nonvanishing within the shaded region, and elsewhere the transition is continuous, controlled by the line of fixed points P_1q_1.

which controls the irrelevance of $T_{\rm RF}$ and the consequent violation of hyperscaling [13]. In the same manner, it may be shown that the latent heat vanishes as $(w_c - w)^{\beta_{\rm RF}}$ as the line Rq_2 is approached from below. Of course, these relations have been established only close to $d = 2$, but, if the topology of the RG flows does not change, they should hold also in three and higher dimensions.

Above the line Rq_2, the flows go to large w beyond the validity of (1)–(3). In addition, the renormalized interfacial tension flows to zero and the mapping between the models breaks down as domains of different topologies proliferate. However, for infinite q the mapping remains exact and the flows go to infinite w. This cannot happen for finite q since this is the percolation limit $K_1/K_2 = 0$, at which w^{-1} is *relevant* [14]. There must therefore exist another line of stable fixed points emerging from P_1, which control the universal continuous transition for large, but finite, values of w and q. It is tempting to conjecture, as indicated by the dashed line in Fig. 1, that this connects onto that found by expansion in powers of $q - q_1$ [8], where q_1 is the point where the exponent α of the pure model changes sign [1]. Our analysis indicates that, at least for $d = 2$, this is the case. In 2D (when $q_1 = 2$, $q_2 = 4$), the shaded region collapses, and for any nonzero w the renormalized interfacial tension, and thus the latent heat, vanish. The flows should be towards the line P_1q_1, with a crossover length which, from Eq. (1), has the form $\xi_X \sim e^{1/2Aw^2}$ and therefore may become very large for weak randomness.

We now turn to our numerical results for $d = 2$. These will be described in detail elsewhere [15]. As shown by Blöte and Nightingale [16] the transfer matrix for the pure q-state Potts model in a strip of width L may be constructed in a basis in which q enters only as a continuous parameter. We have generalized this to the case when the bond strengths u_{ij} are quenched random variables, and the transfer matrices $\mathcal{T}_i$ therefore depend on the row labels i. The size of the transfer matrices grows like 4^L, independent of q, making this a practicable method for larger q.

Starting with some suitable initial vector $\mathbf{v}_0$, the leading Lyapunov exponent is given by [17] $\Lambda_L^0 = \lim_{m\to\infty} \times \frac{1}{m} \ln \|(\prod_{i=1}^m \mathcal{T}_i)\mathbf{v}_0\|$. Higher exponents are found by

iterating a set of vectors $(\mathbf{v}_i)_{j=0}^{k}$, where a given $\mathbf{v}_j$ is orthogonalized to the set $(\mathbf{v}_i)_{i=0}^{j-1}$ after each multiplication by the transfer matrix. The average free energy per site is then $f_L = -\frac{1}{L}\Lambda_L^0$. For a system exhibiting a first-order transition with a bulk correlation length ξ we expect [16] $f_L \sim f_\infty + O(L^{-d}e^{-L/\xi})$, so that $\lambda(L) \equiv \ln[f_L - f_\infty] + d\ln L \sim \text{const} - L/\xi$. In Fig. 2 we show this for various values of q and the randomness strength $R \equiv K_2/K_1$. We see that the randomness changes the transition for $q = 8$ into one with an apparently diverging correlation length. In such a case the amplitude of the finite-size correction has the form [18] $f_L \sim f_\infty - \pi c'/6L^2$, where c' is the *effective* central charge [since f_L is the quenched free energy, in a replica formalism this is the derivative of the central charge $c(n)$ with respect to the number of replicas at $n = 0$]. The value of c' was determined by making parabolic least squares of f_L versus $1/L^2$ [19]. We found the optimum trade-off between statistical errors and a reasonable computation time by taking strips of length $m = 10^5$ and averaging f_L over 100 independent realizations of the randomness for $1 \le L \le 8$, and 3 realizations for $9 \le L \le 12$. Data collection was made for each 200 multiplications by $\mathcal{T}_i$, and the first 2000 iterations of each run were discarded in order to eliminate transients. The parabolic fits were made by including the data points for $L_0 \le L \le 12$, where L_0 must be chosen large enough to justify the finite-size scaling form, and small enough to minimize error bars. From the special cases of the Ising model and percolation it appeared that $L_0 = 3$ is optimal.

For the random-bond Ising model ($q = 2$) with $R = 2$ we found $c' = 0.495 \pm 0.006$, in agreement with the result of de Queiroz [19], $c' = 0.498 \pm 0.003$ using the spin basis, and with the expected value $c' = \frac{1}{2}$. For $q = 3$ and $R = 2$, our result $c' = 0.799 \pm 0.006$ is unable to distinguish between the pure value of $\frac{4}{5}$ and that of $c' \approx 0.8025$ obtained in Ref. [8] by an expansion in $q - 2$. For $q = 4$ the results $c' = 1.003 \pm 0.006$ for $R = 2$ and $c' = 1.010 \pm 0.022$ for $R = 10$ are consistent with each other and the pure value $c = 1$, but for larger q the $R = 2$ results appear to saturate while those for $R = 10$ show a gradual increase: $c' = 1.517 \pm 0.025$ for $q = 8$ and $c' = 3.003 \pm 0.031$ for $q = 64$. Similar values have recently been reported by Picco [20]. However, it should be pointed out that these are also very close to those expected at the percolation point $R \to \infty$. For then the replicated model is the Potts model with q^n states, so that $c' = (\partial/\partial n)c_{\text{Potts}}(q^n)|_{n=0} = (5\sqrt{3}/4\pi)\ln q \approx 0.689 \ln q$; this is confirmed by our transfer matrix calculations [15]. These are also remarkably close to the pure values for $2 \le q \le 4$. We conclude that measurement of the effective central charge does not distinguish well between pure, percolative, and nontrivial random behavior.

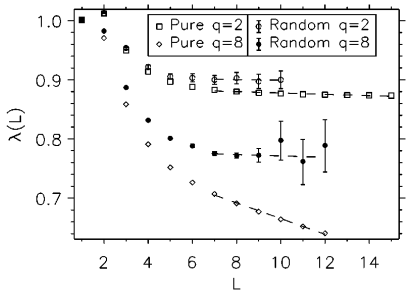

FIG. 2. Plots of $\lambda(L)$, normalized to $\lambda(1) = 1$, showing that bond randomness renders the phase transition second order.

We therefore turned to the magnetic exponent $x_1 = \beta/\nu$. Although this may be determined in principle by adding a ghost site [16], it proved more efficient to exploit duality and relate the spin-spin correlation function $G(m)$ on the strip to the free energy in the presence of a seam of frustrated bonds. Details of this relation and how to implement it in the transfer matrix will be given elsewhere [15]. In pure systems, according to conformal symmetry [21] $G(m)$ decays along the strip as $\exp(-2\pi x_1 m/L)$, so that the difference Δf_L in the free energy per site with and without the seam behaves as $2\pi x_1/L^2$. However, in the random system this difference typically has fluctuations which are $O(m^{-1/2})$. This has the consequence that, while $\Delta f_L \sim \frac{1}{mL}\ln G$ is self-averaging, G is not [22]. In fact, as shown by Ludwig [9], the moments $\overline{G(m)^N}$ (where the overline denotes the quenched average) exhibit *multiscaling;* that is, they scale with dimensions x_N which are not, in general, linear in N. Since conformal symmetry assumes translational invariance, it refers only to such averaged quantities. In practice, we can avoid the lack of self-averaging by performing a cumulant expansion

$$\ln \overline{G^N} = N\overline{\ln G} + \tfrac{1}{2}N^2\overline{(\ln G - \overline{\ln G})^2} + \ldots, \qquad (4)$$

where the higher cumulants may be directly extracted from the statistical fluctuations in Δf_L. For values of q and R which are not too large, this expansion appears to converge well, keeping the first 3 or 4 cumulants. The fact that the higher cumulants are nonzero implies multiscaling. Our values for x_1 are shown in Fig. 3. For $2 < q \le 3$ and $R = 2$ they are in perfect agreement with the predictions of the $(q - 2)$ expansion of Refs. [9,10], and for $q = 3$ and 4 they agree with recent Monte Carlo results of Picco [23]. For larger q the results appear to increase smoothly with q, with $x_1(q = 8) = 0.1415 \pm 0.0036$. Thereafter the cumulant expansion begins to break down.

Although the thermal exponent ν should be related in a similar manner to the first gap $\Lambda_L^1 - \Lambda_L^0$ in the Lyapunov spectrum, for reasons we do not understand this yields results which, if taken literally, appear to violate the bound $\nu \ge 1$ [24]. Instead we have measured ν directly by finite-size scaling of the magnetic correlation length away from T_c, using phenomenological RG methods [25]. For $q = 8$, as shown in Fig. 4, we find clear evidence for fixed points at $R = 1$, with $\nu = \frac{1}{2}$ ($x_T = 0$), and a random fixed point with $\nu_R = 1.01 \pm 0.02$. At $q = 3$ our results are consistent with the perturbative value $\nu_R \approx 1.02$, and for $q = 64$ we find $\nu_R = 1.02 \pm 0.03$.

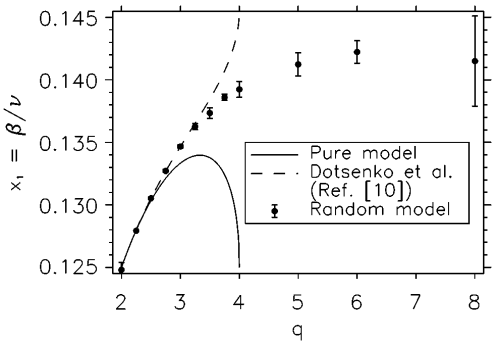

FIG. 3. Magnetic exponent as a function of q, with $R = 2$.

In summary, we have computed the exponents of the 2D random-bond Potts model and shown that while the thermal exponent ν_R is consistent, within error bars, with both the pure Ising value, and with the results of the $(q - 2)$ expansion, the magnetic exponent varies continuously with q in a manner which agrees with this expansion in the region it is expected to be valid, and with a value at $q = 8$ which is quite different from the Ising value of $\frac{1}{8}$. This is in sharp disagreement with the Monte Carlo results of Ref. [6]. One possible reason is that these authors use a nonstandard definition of the order parameter which, while it scales in the same way as the usual one in the pure case, may not when multiscaling is present. Another is that our very long strips are able to accommodate large regions in which all q values of the order parameter are realized, and this may not typically be the case in the square geometries of Ref. [6]. Our results should also be compared with those of Ref. [26], in which it is shown that the random *connectivity* of a Voronoi–Delauney lattice does not modify the first-order nature of the transition. However, in this case it may be shown that the typical random fluctuations in the energy of a region of size L^d are $O(L^{(d-1)/2})$ rather than $O(L^{d/2})$ as in our random-bond model, so they are always smaller than the domain wall energy. Another study [27], in which the transition was observed to be softened, refers to a lattice with random *curvature* which is fractal when embedded in the plane,

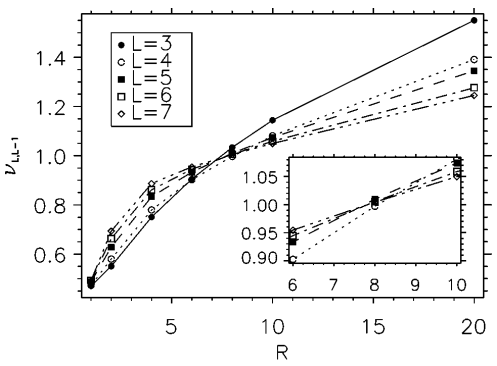

FIG. 4. Values of ν extracted from phenomenological RG for different strip widths and $q = 8$. There is a fixed point at $R \approx 8$ with $\nu \approx 1$ (see inset). Error bars are less than the symbol size.

and to which neither our arguments nor those of Refs. [3,4] directly apply.

We have also given a mapping of the random-bond problem to the RFIM which suggests that for $d > 2$ such systems should exhibit a tricritical point whose thermal exponent is related to those of the RFIM. This picture is quite generic, and, since there are many real 3D systems which undergo first-order transitions, it would be interesting to reexamine the effect of random impurities in such cases.

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