Theoretical Study of the Damping of Collective Excitations in a Bose-Einstein Condensate

W. Vincent Liu*
Theory Group, Department of Physics, University of Texas, Austin, Texas 78712-1081
(Received 7 August 1997)

We study the damping of low-lying collective excitations of condensates in a weakly interacting Bose gas model within the framework of an imaginary time path integral. A general expression of the damping rate has been obtained for both the very low temperature regime and the higher temperature regime. For the latter, the result is new and applicable to recent experiments. Theoretical predictions for the damping rate are compared with the experimental values. [S0031-9007(97)04625-5]

PACS numbers: 03.75.Fi, 11.10.Wx, 67.40.Db

The recent realization of Bose-Einstein condensation (BEC) in dilute atomic vapors [1,2] has opened the door to experimentally study weakly interacting dilute quantum gases for which microscopic theories have been well studied for decades. In particular, it has been shown that the frequencies of the low-lying collective excitations of condensates [3,4] agree excellently with theoretical predictions based on a mean-field theory [5]. The damping in such collective modes has been experimentally discovered by Jin *et al.* [3] and Mewes *et al.* [4]. A very recent experiment [6] has extended the study of low-lying collective excitations of condensates to include higher temperatures in which the damping of these excitations exhibits dramatic temperature dependence. So far, there is no theoretical prediction for it in such a temperature regime.

This paper aims to provide a simple way of calculating the damping rate of collective excitations in the above temperature regime. The experimental setup we consider is a dilute gas of atoms confined in a trap potential. Recently, Chou, Yang, and Yu [7] argued that the local density approximation can be applied into the present problem. In other words, the trap potential may be treated as a slowly varying external potential and enters into the theory only as a modification to the chemical potential. To simplify the theory, let us assume that the trap is absent in deriving the damping rate since we are only interested in finding the leading term for it. In the following, we shall study a model of weakly interacting bosonic particles in an infinite free space using the imaginary time path integral method which allows us to take into account the excitation collision processes that are believed to be the cause of the damping.

In the low momentum and low temperature limit, one usually finds an effective field theory in which only slowly varying degrees of freedom appear explicitly with interactions that include the effects of fast varying fields that have first been integrated out [8]. One version of such an effective field theory has been established for a nonideal Bose gas in the low temperature regime by Popov [9]. The theory treats the system effectively as that of quasiparticles described by two slowly varying real scalar fields $\phi(x)$ and $\sigma(x)$. Here, the $\sigma(x)$ fields describe the density fluctuations, written as $\sigma(x) = n(x) - n_0$ with n_0 the density of the ground state. The $\phi(x)$ fields are the Goldstone field in this theory, appearing originally in the phase of original particle fields. Notice that we have written the four-dimensional Euclidean space-time as $x = (\mathbf{x}, \tau)$ with $\tau = it$ denoting the Euclidean "time." According to Popov, the Euclidean action for such a system can be written by ($\hbar \equiv 1$ henceforth)

$$S[\phi,\sigma] = \int_0^\beta d\tau \int_{-\infty}^{\infty} d^3x \Big\{ i \frac{\partial^2 p}{\partial\mu\partial n}\, \sigma \partial_\tau \phi(x) - \frac{1}{2m}\frac{\partial p}{\partial \mu}[\nabla\phi(x)]^2 - \frac{1}{2}\frac{\partial^2 p}{\partial\mu^2}[\partial_\tau\phi(x)]^2 + \frac{1}{2}\frac{\partial^2 p}{\partial n^2}\sigma^2(x) - \frac{[\nabla\sigma(x)]^2}{8mn_0} - \frac{\sigma(x)[\nabla\phi(x)]^2}{2m} \Big\}, \tag{1}$$

where m is the particle mass and p is the pressure of the system given as the function of the chemical potential μ and the particle density n. Notice that $\phi(x)$ and $\sigma(x)$ are periodic in time τ with period $\beta = 1/(k_B T)$. If $\mathbf{v} = (1/m)\nabla\phi$ is identified with the phonon velocity field, one can easily verify that the action (1) corresponds to a Hamiltonian that is one form of a Landau-Khalatnikov hydrodynamic Hamiltonian [10]. In the low temperature regime such that the noncondensate fraction of particles is very small, the expression of the pressure $p(\mu, n)$ can be well approximated by that of $T = 0$. For a weakly (repulsively) interacting dilute gas, it is given by [11] $p = \mu n - \frac{1}{2} t_0 n^2$, where $t_0 = 4\pi a/m$ in the hard-sphere approximation with a the s-wave scattering length. It follows that $\partial^2 p/\partial\mu\partial n = 1$, $\partial p/\partial\mu = n_0$, $\partial^2 p/\partial\mu^2 = 0$, and $\partial^2 p/\partial n^2 = -t_0$.

For simplicity, we write ϕ and σ fields into a real scalar doublet as $\Phi^\dagger = (\phi, \sigma)$. The Green's functions (or propagators) are defined by the matrix,

$$G(x - x') = \langle T\{\Phi(x)\Phi^\dagger(x')\}\rangle = \frac{\int [\prod_x d\phi(x) d\sigma(x)] \Phi(x)\Phi^\dagger(x') e^{S[\phi,\sigma]}}{\int [\prod_x d\phi(x) d\sigma(x)] e^{S[\phi,\sigma]}},$$

with T denoting a time-ordered product. The Fourier transform of $G(x)$ is defined through

$$G(x) = \frac{1}{\beta(2\pi)^3} \sum_\nu \int d^3k G(k) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega_\nu \tau)}, \quad (2)$$

where the notation $k \equiv (\mathbf{k}, i\omega_\nu)$ is understood and Matsubara frequencies $\omega_\nu = 2\pi\nu/\beta$ ($\nu = 0, \pm 1, \pm 2, \ldots$). The quadratic part of the action S can be written into the form of $S_{\text{QUAD}} = -\frac{1}{2}\int d^4x d^4x' \Phi^\dagger(x) \mathcal{D}(x,x') \Phi(x')$ and the free Green's function $G_0(k)$ is equal to the inverse of the matrix $\mathcal{D}$. Thus, we find

$$G_0^{-1}(k) = \begin{pmatrix} \frac{n_0}{m} k^2 & \omega_\nu \\ -\omega_\nu & t_0 + \frac{k^2}{4mn_0} \end{pmatrix}. \quad (3)$$

It follows that

$$G_0(k) = \begin{pmatrix} \frac{t_0 + k^2/(4mn_0)}{\omega_\nu^2 + \epsilon^2(\mathbf{k})} & \frac{-\omega_\nu}{\omega_\nu^2 + \epsilon^2(\mathbf{k})} \\ \frac{\omega_\nu}{\omega_\nu^2 + \epsilon^2(\mathbf{k})} & \frac{(n_0/m)k^2}{\omega_\nu^2 + \epsilon^2(\mathbf{k})} \end{pmatrix}, \quad (4)$$

where the spectrum $\epsilon(\mathbf{k}) = \sqrt{(\frac{k^2}{2m})^2 + c^2k^2}$ with $c \equiv \sqrt{t_0 n_0/m}$. The propagators G_0 are represented by Feynman diagrams in Fig. 1. Further, the cubic term of the action (1) describes the interaction of three excitations that are known as phonons in the low momentum region, giving rise to a vertex of $\delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)\delta_{\nu_1+\nu_2+\nu_3,0}[(\mathbf{k}_1 \cdot \mathbf{k}_2)/m]$ represented by the last diagram of Fig. 1. Obviously, it can be treated as perturbation in the low momentum limit.

The spectrum of collective excitations is given by the poles of the exact Green's function [12] $G(k) = G_0(k) + G_0(k)\Pi(k)G(k)$, where $\Pi(k)$ denotes the matrix of the self-energy parts. It follows that the spectrum is determined by

$$\det G^{-1}(k) = \det[G_0^{-1}(k) - \Pi(k)] = 0. \quad (5)$$

Now that we are working in the imaginary time formalism, let us make the analytical continuation $i\omega_\nu \to \omega + i\eta$ ($\eta \equiv 0^+$) after the Matsubara frequency sum and write $\omega = \mathrm{Re}\omega - i\gamma(\mathbf{k})$ with γ denoting the damping rate. Then, keeping only the terms up to one-loop order [13], we find from Eqs. (3) and (5) that

$$\gamma(\mathbf{k}) = \frac{1}{2\mathrm{Re}\omega}\left[\left(t_0 + \frac{k^2}{4mn_0}\right)\mathrm{Im}\Pi_{\phi\phi}(\mathbf{k}, \omega + i\eta) + \frac{n_0k^2}{m}\mathrm{Im}\Pi_{\sigma\sigma}(\mathbf{k}, \omega + i\eta)\right] - \mathrm{Re}\Pi_{\phi\sigma}(\mathbf{k}, \omega + i\eta), \quad (6)$$

with the understanding that all ω's are replaced with $\epsilon(\mathbf{k})$ after the analytical continuation. Up to one-loop order, there are six diagrams (see Fig. 2) for the self-energy matrix $\Pi(\mathbf{k}, \omega)$, but only the last five Figs. 2(b)–2(f) contribute to the damping. After collecting contributions from all related diagrams, we have

$$\gamma(\mathbf{k}) = \gamma_1(\mathbf{k}) + \gamma_2(\mathbf{k}), \quad (7)$$

with

$$\gamma_1(\mathbf{k}) = \frac{1}{32\pi^2}\int d^3k' \delta(\epsilon(\mathbf{k}) - \epsilon(\mathbf{k}') - \epsilon(\mathbf{k} - \mathbf{k}'))[f(\epsilon(\mathbf{k}')) - f(-\epsilon(\mathbf{k} - \mathbf{k}'))] \times \left\{\frac{(\mathbf{k} - \mathbf{k}')^2(\mathbf{k}\cdot\mathbf{k}')^2\epsilon(\mathbf{k}')\epsilon(\mathbf{k})}{2mn_0k^2k'^2\epsilon(\mathbf{k} - \mathbf{k}')} + \frac{(\mathbf{k}\cdot\mathbf{k}')[\mathbf{k}\cdot(\mathbf{k} - \mathbf{k}')]\epsilon(\mathbf{k})}{2mn_0k^2} + \frac{k^2[\mathbf{k}'\cdot(\mathbf{k} - \mathbf{k}')]^2\epsilon(\mathbf{k}')\epsilon(\mathbf{k} - \mathbf{k}')}{4mn_0k'^2(\mathbf{k} - \mathbf{k}')^2\epsilon(\mathbf{k})} + \frac{[\mathbf{k}'\cdot(\mathbf{k} - \mathbf{k}')](\mathbf{k}\cdot\mathbf{k}')\epsilon(\mathbf{k}')}{mn_0k'^2}\right\} \quad (8)$$

and

$$\gamma_2(\mathbf{k}) = \frac{1}{32\pi^2}\int d^3k' \delta(\epsilon(\mathbf{k}) + \epsilon(\mathbf{k}') - \epsilon(\mathbf{k} + \mathbf{k}'))[f(\epsilon(\mathbf{k}')) - f(\epsilon(\mathbf{k} + \mathbf{k}'))] \times \left\{\frac{\epsilon(\mathbf{k})}{2mn_0}\left[\frac{k'^2[\mathbf{k}\cdot(\mathbf{k} + \mathbf{k}')]^2\epsilon(\mathbf{k} + \mathbf{k}')}{k^2(\mathbf{k} + \mathbf{k}')^2\epsilon(\mathbf{k}')} + \frac{(\mathbf{k} + \mathbf{k}')^2(\mathbf{k}\cdot\mathbf{k}')^2\epsilon(\mathbf{k}')}{k^2k'^2\epsilon(\mathbf{k} + \mathbf{k}')}\right] + \frac{(\mathbf{k}\cdot\mathbf{k}')[\mathbf{k}\cdot(\mathbf{k} + \mathbf{k}')]\epsilon(\mathbf{k})}{mn_0k^2} + \frac{k^2[\mathbf{k}'\cdot(\mathbf{k} + \mathbf{k}')]^2\epsilon(\mathbf{k}')\epsilon(\mathbf{k} + \mathbf{k}')}{2mn_0k'^2(\mathbf{k} + \mathbf{k}')^2\epsilon(\mathbf{k})} + \frac{[\mathbf{k}'\cdot(\mathbf{k} + \mathbf{k}')]}{mn_0} \times \left[\frac{(\mathbf{k}\cdot\mathbf{k}')\epsilon(\mathbf{k}')}{k'^2} + \frac{[\mathbf{k}\cdot(\mathbf{k} + \mathbf{k}')]\epsilon(\mathbf{k} + \mathbf{k}')}{(\mathbf{k} + \mathbf{k}')^2}\right]\right\}, \quad (9)$$

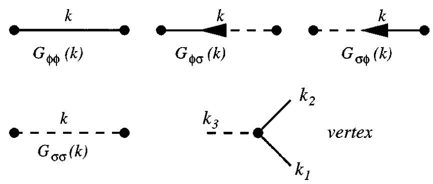

FIG. 1. Diagrams for propagators and vertex.

where $f(\epsilon) = 1/[\exp(\beta\epsilon) - 1]$.

If $T = 0$, one can easily verify that, for small k ($ck \ll n_0 t_0$),

$$\gamma_{T=0}(\mathbf{k}) = \frac{3k^5}{640\pi m n_0} \tag{10}$$

which is the well-known Beliaev's result [14]. The product $n_0 t_0$ characterizes the strength of particle interactions.

For $T \neq 0$ and small k such that $ck \ll k_B T$ and $ck \ll n_0 t_0$, we find that the damping rate to the lowest order in k is determined by

$$\gamma(\mathbf{k}) = \frac{ckk_0^5}{16\pi m n_0 k_B T}\, I\left(\frac{n_0 t_0}{k_B T}\right), \tag{11}$$

where $k_0 \equiv \sqrt{m n_0 t_0}$ and the function $I(x)$ is given by

$$I(x) = \frac{\pi^2}{6x^3} + \int_0^\infty d\xi \left[\frac{2\xi^2}{(1+\xi^2)^{3/2}} + \frac{3}{2(1+\xi^2)} - \frac{2}{(1+\xi^2)^2}\right] \frac{\xi^2 e^{x\xi}}{(e^{x\xi}-1)^2}.$$

If the limit $k_B T/n_0 t_0 \to 0$ is taken, Eq. (11) reduces to the familiar form

$$\gamma(\mathbf{k}) = \frac{3\pi^3 k (k_B T)^4}{40 m n_0 c^4} \tag{12}$$

which was given by Hohenberg and Martin [15]. Although, we shall see in the following paragraph that Eq. (12) is invalid in the recent experimental temperature regime. In fact, both expressions (10) and (12) were already derived by Popov in Ref. [9] from the effective action (1). But, to the best of my knowledge, the general expression (11) is first obtained here. It is valid as long as the system is in the low temperature region such that the number of particles in the excited states is much less than that in the condensate.

The damping of collective excitations in BEC has been measured in the dilute atomic vapors of both ^{87}Rb and sodium at JILA and MIT, respectively [3,4,6]. For the experiment of ^{87}Rb at JILA, the condensates are produced in a trap with frequencies of $\nu_r = 129$ Hz radially and $\nu_z = 365$ Hz axially. For a typical condensate of $N_{\mathrm{BEC}} = 4500$ atoms, the density of a condensate can be estimated as $n_0 \approx 1.4 \times 10^{14}$ cm^{-3} [16]. Furthermore, the scattering length for a ^{87}Rb vapor may be taken roughly as $a = 103$ bohrs [17]. It turns out that the typical interaction energy $n_0 t_0 \approx 57$ nK. Therefore, the damping rate is expected to be given by Eq. (11) instead of Eq. (12) in the temperature region (30, 300) nK, where it was measured (see Figs. 1 and 3 of Ref. [6]). We also checked that for the lowest excitation mode ($m = 2$) both $ck/(k_B T)$ and $ck/(n_0 t_0)$ are less than 0.2 for any T above 50 nK. Figure 3 shows theoretical predictions for the damping rate of collective excitations in comparison with the experimental data from Ref. [6]. The theoretical curve (b) seems to fit the experiment well. Nevertheless, the damping rate is varying with the density of condensates, and how to determine a proper average density n_0 for a trapped Bose condensate shall not be discussed here. Next, let us check whether the above experiment is in the low temperature region. For $T < 100$ nK, we read off from Fig. 1 of Ref. [6] that the corresponding $T' < 0.6$ (where $T' \equiv T/T_c$), and the condensate fraction N_{BEC}/N is greater than or about 80%. Hence, the higher order corrections to Eq. (11) due to finite temperature effects can only be estimated from the calculation of Feynman diagrams less than $(1 - N_{\mathrm{BEC}}/N)^2 = 4\%$. On the other hand, Ref. [6] shows that more than 50% of particles reside in noncondensates when $T' > 0.8$ and, correspondingly, $T > 175$ nK roughly. This means that Eq. (11) is invalid in that temperature regime. Instead, the damping

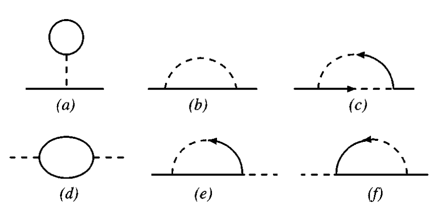

FIG. 2. One-loop diagrams for the self-energy matrix $\Pi(k)$.

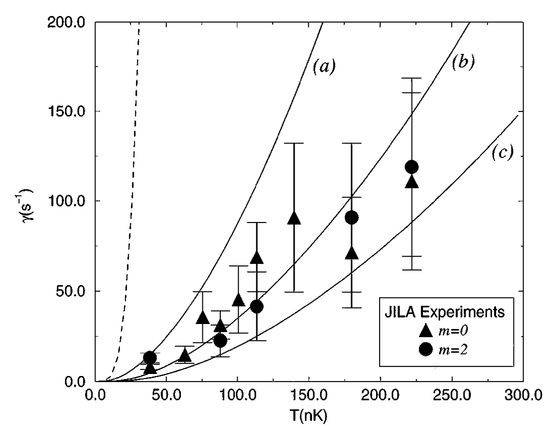

FIG. 3. The damping rate of collective excitations in a ^{87}Rb atomic gas. The three solid lines indicate theoretical predictions from Eq. (11). For those, the excitation frequency is taken as $\omega/2\pi = 1.4\nu_r = 180.6$ Hz corresponding to the mode $m = 2$, and the condensate densities are (a) $n_0 = 1.0 \times 10^{14}$ cm^{-3}, (b) $n_0 = 2.0 \times 10^{14}$ cm^{-3}, and (c) $n_0 = 3.0 \times 10^{14}$ cm^{-3}. The dashed line is the prediction of Eq. (12) with the same excitation frequency and $n_0 = 1.0 \times 10^{14}$ cm^{-3}. Two discrete curves are replotted from the data of Ref. [6]. (The experimental data are the courtesy of M. R. Matthews.)

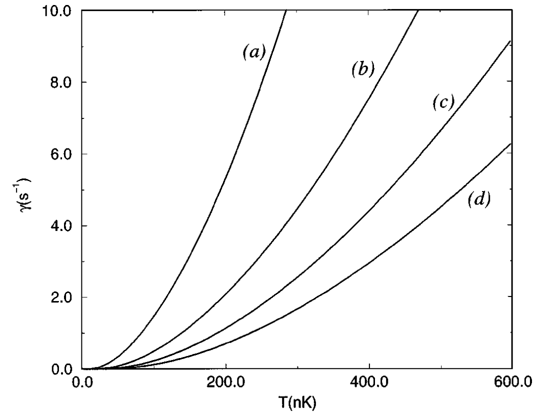

FIG. 4. The damping rate of collective excitations in a sodium atomic gas. All solid lines are plotted according to Eq. (11). The excitation frequency is 30 Hz and the condensate densities are (a) $n_0 = 1.0 \times 10^{14}$ cm^{-3}, (b) $n_0 = 2.0 \times 10^{14}$ cm^{-3}, (c) $n_0 = 3.0 \times 10^{14}$ cm^{-3}, and (d) $n_0 = 4.0 \times 10^{14}$ cm^{-3}.

rate was found to be linear in T for higher temperature (below T_c) in the theory of Szepfalusy and Kondor [18].

In Fig. 4, we plot the damping rate of the collective excitation of frequency 30 Hz for the sodium gas system of Mewes *et al.* [4] using the s-wave scattering length $a = 65$ bohrs [19]. The decay time of 250(40) ms was found experimentally when a nearly pure condensate was formed at the temperature $T \approx 0.5T_c$ [20]. The Bose-Einstein transition temperature is determined theoretically according to $T_c = (\hbar\bar{\omega}/k_B)(N/1.202)^{1/3}$ [19,21] with $\bar{\omega}$ the geometric mean of the harmonic trap frequencies $\bar{\omega} = (\omega_x\omega_y\omega_z)^{1/3}$. For the trap used in Ref. [4] is of frequencies of 250 Hz radially and 19 Hz axially, and typically holds a total number $N \approx 5 \times 10^6$ of atoms when the decay time is measured, we have $T_c \simeq 800$ nK. For a typical condensate density $n_0 = 3 \times 10^{14}$ cm^{-3} [19], we can read off from the curve (c) of Fig. 4 that $\gamma \simeq 4.4$ s^{-1} for temperature $T = 400$ nK. This damping rate corresponds to a decay time of about 230 ms that agrees very well with the experimental value. Also, since at such a temperature the condensate fraction of atoms is around 90%, higher order corrections to γ due to the finite temperature effect alone are quite small (<1%). Also, we checked that both $ck/(k_BT)$ and $ck/(n_0t_0)$ are less than 1%. That is to say, the system is well in the low momentum region.

In conclusion, this paper calculates the damping rate of collective excitations for a dilute Bose gas model in a temperature regime where theoretical predictions did not exist previously. Although the model has ignored the contribution of the trap and has required that the condensate be homogeneous, it produces results for the damping of collective excitations that are in good agreement with the experiments. Our study also reveals that the damping is due mainly to the process that one excitation absorbs a phonon transferring into another at finite temperature. Fully understanding it requires a complete analysis to take into account the contribution of the trap and the inhomogeneity of the system.

I am indebted to Professor S. Weinberg for guidance and for carefully reading the manuscript. I am grateful to E. Cornell and W. Ketterle for very helpful instructions on the experiments, to M.R. Matthews for sending me the experimental data, to G. Ordonez for helpful conversations, and to Professor E.C.G. Sudarshan for illuminating discussions on BEC. This work is supported by NSF Grant No. PHY-9511632 and the Robert A. Welch Foundation.

*Electronic address: liu@physics.utexas.edu

[1] M.H. Anderson *et al.*, Science **269**, 198 (1995).
[2] K.B. Davis *et al.*, Phys. Rev. Lett. **75**, 3969 (1995).
[3] D.S. Jin *et al.*, Phys. Rev. Lett. **77**, 420 (1996).
[4] M.-O. Mewes *et al.*, Phys. Rev. Lett. **77**, 988 (1996).
[5] See, for example, S. Stringari, Phys. Rev. Lett. **77**, 2360 (1996); M. Edwards *et al.*, Phys. Rev. Lett. **77**, 1671 (1996).
[6] D.S. Jin *et al.*, Phys. Rev. Lett. **78**, 764 (1997).
[7] T.T. Chou, C.N. Yang, and L.H. Yu, Phys. Rev. A **53**, 4257 (1996).
[8] S. Weinberg, *The Quantum Theory of Fields II: Modern Applications* (Cambridge University Press, Cambridge, England, 1996), Chaps. 19 and 21.
[9] V.N. Popov, Theor. Math. Phys. **11**, 478 (1972); V.N. Popov, *Functional Integrals in Quantum Field Theory and Statistical Physics* (Reidel, Dordrecht, 1983), pp. 148–160.
[10] I.M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (Addison-Wesley, Redwood City, Calif., 1989), Chap. 7.
[11] This is given in Ref. [9]. See also, for example, K. Huang, *Statistical Mechanics* (Wiley, New York, 1987), Chap. 12.
[12] A. Fetter and J.D. Walecka, *Quantum Theory of Many Particle Systems* (McGraw-Hill, New York, 1971).
[13] We assume that proper renormalization procedures have been carried out so that no diagram for the matrix Π diverges.
[14] S.T. Beliaev, Sov. Phys. JETP **7**, 289 (1958); **34**, 299 (1958).
[15] P.C. Hohenberg and P.C. Martin, Ann. Phys. (N.Y.) **34**, 291 (1965).
[16] G. Baym and C.J. Pethick, Phys. Rev. Lett. **76**, 6 (1996).
[17] E. Cornell (private communication).
[18] P. Szépfalusy and I. Kondor, Ann. Phys. (N.Y.) **82**, 1 (1974).
[19] M.-O. Mewes *et al.*, Phys. Rev. Lett. **77**, 416 (1996).
[20] W. Ketterle pointed out to me that a "nearly pure condensate" meant that the condensate fraction of atoms was greater than or about 90%. Based on this condition and $N_{\mathrm{BEC}}/N = 1 - (T/T_c)^3$ [21], he argued that the absolute temperature was about $0.5T_c$.
[21] S.R. de Groot, G.J. Hooyman, and C.A. ten Seldam, Proc. R. Soc. London A **203**, 266 (1950).