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Multisoliton Solutions of the Complex Ginzburg-Landau Equation

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We present novel stable solutions which are soliton pairs and trains of the 1D complex Ginzburg-Landau equation (CGLE), and analyze them. We propose that the distance between the pulses and the phase difference between them is defined by energy and momentum balance equations. We present a two-dimensional phase plane ("interaction plane") for analyzing the stability properties and general dynamics of two-soliton solutions of the CGLE. [S0031-9007(97)04655-3]

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The emergence of stable spatiotemporal patterns in a variety of physical situations may be modeled through the well-known complex Ginzburg-Landau equation (CGLE). The CGLE is the basic model which describes nonlinear phenomena far from equilibrium [1]. It describes, for example, open flow motions [2], traveling waves in binary fluid mixtures [3], and spatially extended nonequilibrium systems [4,5]. In optics, it is useful in analyzing optical transmission lines [6,7], passively mode-locked fiber lasers [8,9], and spatial optical solitons [10]. In each case, the problem of the interaction of two, individually stable, juxtaposed elementary coherent structures (i.e., solitons) is crucial for understanding the general behavior of the system [11,12].

Stable pulse-like solutions of the quintic CGLE have been found by Thual and Fauve [13]. Minimal requirements for their stability have been obtained in [14]. In the conservative limit, these solutions can be considered as perturbations of the nonlinear Schrödinger equation (NLSE) solitons [15]. The continuous transition of these solutions from the conservative limit to the gradient limit in the parameter space of the CGLE has been studied in Ref. [16]. Although the dynamical properties of these pulselike solutions, their collisions and interactions are different from those of solitons of integrable systems, they have been called "solitons" in a number of works. We follow this tradition, and also call them "solitons" or "soliton solutions."

For the nonlinear Schrödinger equation, two solitons have zero binding energy. Hence, any nonlinear superposition of two solitons is neutrally stable, and can be made unstable with a very small perturbation. On the other hand, for the NLSE, there is no stationary solution in the form of two solitons with equal amplitudes and velocities and with a fixed separation. Frequently, real systems are not described by integrable equations (e.g., the NLSE), but by Hamiltonian generalizations of the NLSE. For these systems, the interaction between the pulses becomes inelastic, so that two-soliton solutions of the perturbed NLSE (when they exist) are unstable due to the energy exchange between the pulses [17]. The situation changes completely for nonconservative systems. Each soliton then has its own internal balance of energy which maintains its constant amplitude. Fixing the amplitudes effectively reduces the number of degrees of freedom in the system of two solitons and can make it stable. Bound states of two solitons in these systems were first analyzed by Malomed [18]. Using standard perturbation analysis for soliton interaction, he showed that stationary solutions in the form of bound states of two solitons, which are in-phase or out-of-phase, may exist. We also confirm that they do exist. However, careful numerical analysis shows that these types of soliton bound states are unstable [19].

In this work we report the discovery of quintic CGLE *stable* two and more soliton solutions with a $\pi/2$ phase difference between them. We propose using a 2D space (using distance and phase difference) to analyze the dynamics of the two-soliton system, and we show that this space describes the system adequately. This method allows us to find the bound states, analyze their stability, and investigate their global dynamics. The quintic CGLE describes situations where there is a single transverse (or temporal) coordinate (see, e.g., [5,17]):

$$i\psi_\xi + \left(\frac{D}{2} - i\beta\right)\psi_{\tau\tau} + (1 - i\epsilon)|\psi|^2\psi + (\nu - i\mu)|\psi|^4\psi = i\delta\psi\,, \tag{1}$$

where τ is the retarded time, ξ is the propagation distance, δ, β, ϵ, μ, and ν are real constants, ψ is a complex field, and one can always set $D = \pm 1$. We use $D = +1$ (i.e., anomalous dispersion, or self-focusing regime of the corresponding NLS equation).

Bright-soliton solutions of the CGLE form a discrete set [5,22], so that, if the values of the parameters of the equation are specified, then the amplitude and width of the soliton are fixed. There may be many solutions for each parameter set [4,21], but each solution has a fixed amplitude and phase profile. Thus CGLE solitons differ qualitatively from those of Hamiltonian systems, where all bright soliton solutions are always members of a family of solutions with variable amplitude. There are exceptions when there is a certain relation between the parameters and new symmetries appear [22], and for dark-soliton-type solutions of the cubic CGLE [5,23], but these are very special cases. The physical reason for the above fact is that, in contrast to the NLSE and its Hamiltonian generalizations, solitons of the CGLE arise as a result of a balance between the nonlinearity and dispersion on the one hand, and between the gain and loss on the other hand. Either of these, independently, would define a family of solutions, but imposing both simultaneously usually gives a fixed solution.

The fact that the soliton parameters are fixed implies that, during the interaction of two solitons, basically only two parameters may change: their separation ρ and the phase difference ϕ between them. Thus the phase space here is truly 2D, and we may analyze the bound states formed of two solitons, their stability and their global dynamics in this 2D space, which we call the "interaction plane." The possibility of this reduction in the number of degrees of freedom is a unique feature of systems with gain and loss. It does not apply for nonintegrable Hamiltonian systems, where the amplitudes of the solitons can also change, and therefore more sources of instability of the bound states appear [17].

The CGLE has no known conserved quantities. Instead, the energy associated with solutions ψ is $Q = \int_{-\infty}^{\infty} |\psi|^2\, d\tau$, and its rate of change with respect to ξ is [17]

$$\frac{d}{d\xi} Q = F[\psi]\,, \tag{2}$$

where the functional $F[\psi]$ is given by

$$F[\psi] = 2\int_{-\infty}^{\infty} [\delta|\psi|^2 + \epsilon|\psi|^4 + \mu|\psi|^6 - \beta|\psi_\tau|^2]\, d\tau\,. \tag{3}$$

Similarly, the momentum is $M = \mathrm{Im}(\int_{-\infty}^{\infty} \psi_\tau^* \psi\, d\tau)$, and its rate of change is defined by

$$\frac{d}{d\xi} M = J[\psi]\,, \tag{4}$$

where the real functional $J[\psi]$ is given by

$$J[\psi] = 2\,\mathrm{Im}\int_{-\infty}^{\infty} [(\delta + \epsilon|\psi|^2 + \mu|\psi|^4)\psi + \beta\psi_{\tau\tau}] \times \psi_\tau^*\, d\tau\,. \tag{5}$$

By definition, this functional is the force acting on a soliton along the τ axis. There are only two rate equations, viz. (2) and (4), which can be derived for the CGLE. Higher order functionals do not exist.

We seek stationary solutions so the energy and momentum do not change, and the corresponding solutions must satisfy the set of two equations

$$F[\psi] = 0, \qquad J[\psi] = 0\,. \tag{6}$$

The first identity indicates the necessary balance that must exist between losses and gain for any stationary solution, while the second guarantees a balance between the transverse forces acting on solitons. Trivially, $J[\psi] = 0$ for any symmetric $[\psi(\tau) = \pm\psi(-\tau)]$ solution, but $J[\psi]$ may also be zero for other solutions (which can have nonzero velocity).

Given the equation coefficients, we call the corresponding "plain" soliton solution $\psi_0(\tau)$. The bound solution of two plain solitons is well approximated by

$$\psi(\tau) = \psi_0(\tau - \rho/2) + \psi_0(\tau + \rho/2)\exp(i\phi)\,, \tag{7}$$

where the values of ρ and ϕ are those which satisfy Eqs. (6). For very rough estimates, these calculations can be done using simple trial functions for $\psi_0(\tau)$. However, since we have exact numerical soliton solutions, we use them to find the zeros of F and J numerically.

The zeros of these functionals, in the interval $0.4 < \rho < 4$, are presented on the interaction plane in Fig. 1 for the parameters written in the figure. The separation ρ must be of the same order as, but larger than, the width of a single soliton (indicated by a dashed circle in the figure). Smaller ρ correspond to merging of

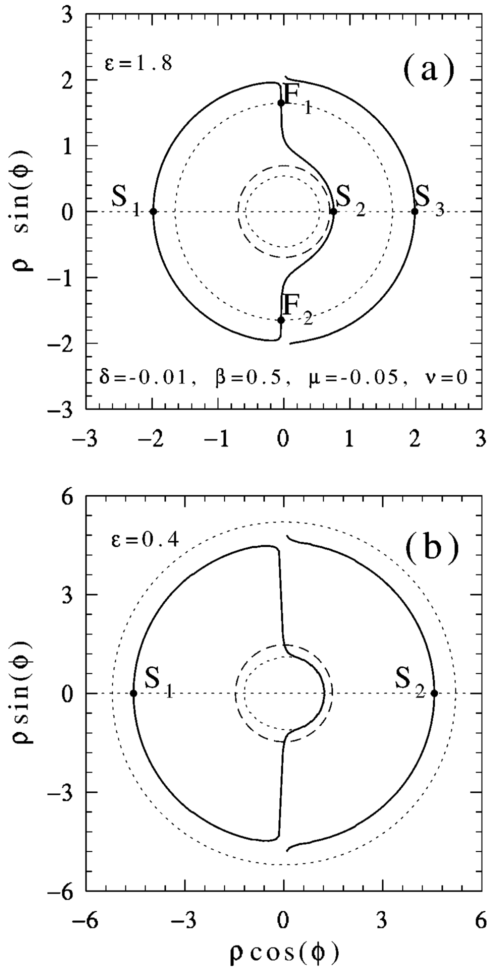

FIG. 1. Zeros of F (solid lines) and J (dotted lines) on the interaction plane for (a) $\epsilon = 1.8$ and (b) $\epsilon = 0.4$. Other parameters are the same for both planes and are shown in part (a) of the figure. The points of intersection of the solid curves with the dotted ones correspond to bound states of two solitons. They are shown as bold points. The radius of the dashed circle indicates the full width at half maximum of a single soliton.

the solitons and at larger ρ the interaction between the solitons is too weak. The solid lines in Fig. 1 show the locus of points where $F[\psi] = 0$, while the dotted lines show those where $J[\psi] = 0$. It can be seen from this figure that the functional $F[\psi]$ has two zeros in the interval $-\pi/2 < \phi < \pi/2$, but only one in the interval $\pi/2 < \phi < 3\pi/2$.

$J[\psi]$ is zero on the horizontal axis of the interaction plane, so every intersection of a solid curve with the horizontal axis corresponds to a two-soliton bound state. There are three examples of this type of bound state in Fig. 1(a), viz. $S_i, i = 1, 2, 3$. For these, the component solitons are in or out of phase. $J[\psi]$ also has zeros along two almost circular arcs. The intersections of the outer circle with the solid curve (points F_1 and F_2) correspond to the new bound states where the phase difference between the solitons is close to $\pi/2$. The phase profile of the solution is necessarily asymmetric, due to this phase difference. For $\epsilon = 0.4$ only two singular points are predicted: $S_{1,2}$ at $\phi = \pi$ and 0, respectively.

The above predictions have been numerically confirmed by solving the propagation equation. The general dynamics of the interaction of two solitons can be described using just the interaction plane. An initial condition (7), in the form of two stable solitons with arbitrary separation (ρ) and phase difference (ϕ), will result in a trajectory on this plane. Bound states are the singular points of this plane, with the type of singular point defining the stability of the state. Figure 2 gives two examples of these numerical simulations, corresponding to Fig. 1. Figure 2(a) indicates that, for the given parameters, there are five singular points. Within the accuracy of the method, these

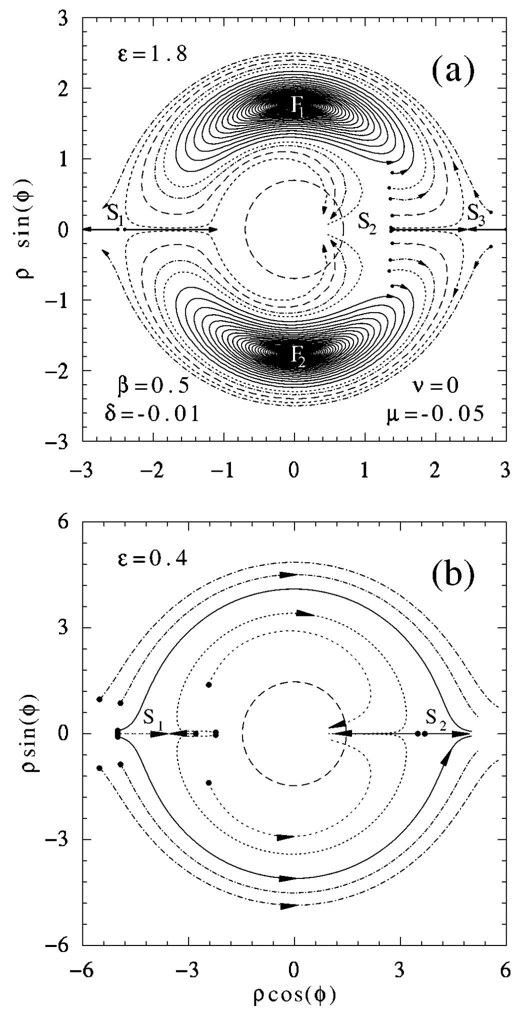

FIG. 2. Trajectories showing the evolution of two-soliton solutions on the interaction plane for the same parameters as those in Fig. 1. The five singular points in (a) correspond to the five bound states depicted in Fig. 1(a). Only two of them (F_1 and F_2) are stable. The two singular points in (b) (S_1 and S_2) are unstable. The central part of the figure, where ρ is less than a single soliton width, does not describe a valid bound state. Trajectories converging to the center describe the merging of two solitons.

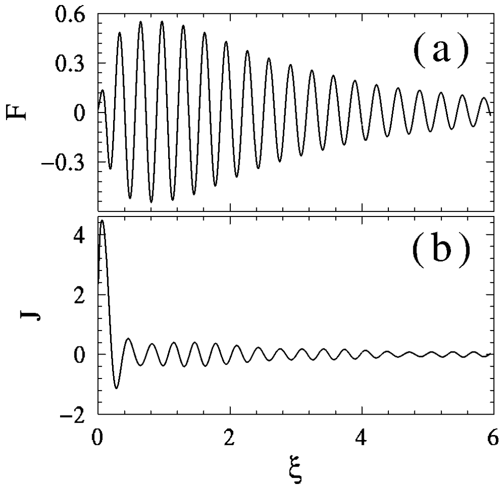

FIG. 3. Oscillations of the functionals (a) F and (b) J in the process of convergence of the initial condition (7) with $\rho = 1.8$ and $\phi = \pi/2$ into a bound state of two solitons. The parameters are $\epsilon = 1.8$, $\delta = -0.01$, $\beta = 0.5$, $\mu = -0.05$, and $\nu = 0$

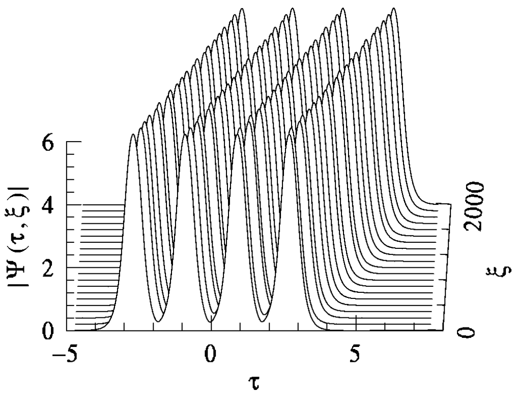

FIG. 4. Stable propagation of a four-soliton bound state. The equation parameters are the same as those in Fig. 3.

coincide with the solutions which are found above using the balance equation. Three of these singular points (S_1, S_2, and S_3) are saddles, with the phase difference between the solitons being zero or π. Clearly, these are unstable bound states of two solitons. In addition, there are two symmetrically located stable foci (F_1 and F_2), and these correspond to stable bound states of two solitons in quadrature, i.e., their phase difference is $\pm\pi/2$. These are the bound states with asymmetric phase profiles predicted above. The spectra are also asymmetric due to the phase asymmetry, as expected for this type of solution [20]. A consequence of this asymmetry is that a two-soliton solution moves with a constant velocity. We should note that asymmetric bound states are not always stable. Changing the parameters in Eq. (1) may convert stable foci into centers (or elliptic points) and further into unstable foci. They can even disappear, as evidenced in Fig. 2(b), where only two singular points (which are saddles) exist, in full agreement with the predictions of the balance equations [see Fig. 1(b)].

We can see now that, physically, the existence of the two-soliton solutions is the result, firstly, of the balance between gain and loss and, secondly, of the balance between forces along the τ axis, which act on the soliton pair. The interplay between the two gives the actual distance and phase difference between the solitons in the bound state.

We should stress that the profile of each single soliton is hardly modified at all by the interaction. The field amplitude and phase of a two-soliton solution and the initial condition which consists of two solitons which are $\pi/2$ out of phase and which have a separation of $\rho = 1.8$ are hardly distinguishable, although they are not identical. It takes a while for the solution to evolve to the bound state, and the process of convergence to the final state of a two-soliton solution can be clearly seen in Fig. 3. Both J and F oscillate with exponentially decaying amplitudes before converging to zero. This indicates that the bound state has been achieved. During this transition, the changes of energy Q are less than 0.03%. Note that, in principle, the nature of the fixed points could be investigated analytically using certain trial functions for single solitons in the pair. However, unavoidable inaccuracies with this type of approximation may cause serious errors in the stability analysis (see an example in [24]).

As a consequence of the existence of two-soliton solutions, three- and more soliton solutions also exist. An example of a multisoliton solution is shown in Fig. 4. As a result of the above-mentioned asymmetry, multisoliton solutions are also asymmetric and move with the same constant velocity along the τ axis. Periodic solutions of the CGLE can clearly be constructed this way, and then the whole train will move with a constant velocity. Schöpf and Kramer [25] were the first to discuss periodic solutions of the CGLE, and in fact their numerical results show that the periodic train has a small transverse velocity [see Fig. 2(a) of [25]]. Note that the approximate analytic solution obtained in that work has zero velocity and does not describe the numerics in Fig. 2(a).

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