Top-quark Pole Mass

Martin C. Smith and Scott S. Willenbrock
Department of Physics, University of Illinois, 1110 West Green Street, Urbana, Illinois 61801
(Received 17 December 1996)

The top quark decays more quickly than the strong-interaction time scale, $\Lambda_{\rm QCD}^{-1}$, and might be expected to escape the effects of nonperturbative QCD. Nevertheless, the top-quark pole mass, like the pole mass of a stable heavy quark, is ambiguous by an amount proportional to $\Lambda_{\rm QCD}$. [S0031-9007(97)04700-5]

PACS numbers: 14.65.Ha, 12.38.Cy, 12.38.Lg

The mass of the recently discovered top quark [1] has been measured with impressive accuracy, $m_t = 175.6 \pm 5.5$ GeV [2], by the CDF and D0 experiments at the Fermilab Tevatron. The uncertainty will be reduced even further, to perhaps 1–2 GeV, with additional running at the Tevatron [3], or at the CERN Large Hadron Collider [4]. High-energy e^+e^- [5] or $\mu^+\mu^-$ [6] colliders operating at the $t\bar{t}$ threshold hold the promise of yet more precise measurements of m_t, to 200 MeV or even better.

With such increasingly precise measurements on the horizon, it is important to have a firm grasp of exactly what is meant by the top-quark mass. Thus far the top-quark mass has been experimentally defined by the position of the peak in the invariant-mass distribution of the top quark's decay products, a W boson and a b-quark jet [2]. This closely corresponds to the pole mass of the top quark, defined as the real part of the pole in the perturbative top-quark propagator. The perturbative propagator of a top quark with four-momentum p has a pole at the complex position $\sqrt{p^2} = m_{\rm pole} - \frac{i}{2}\Gamma$, and yields a peak in the Wb invariant-mass distribution (for experimentally accessible real values of p) when $\sqrt{p^2} \approx m_{\rm pole}$. The extent to which this correspondence continues to hold beyond perturbation theory is one of the topics of this paper.

The pole mass of a stable quark is well defined in the context of finite-order perturbation theory [7]. However, the all-order resummation of a certain class of diagrams, associated with "infrared renormalons," indicates that the pole mass of a stable heavy quark (heavy here means $m \gg \Lambda_{\rm QCD}$) is ambiguous by an amount proportional to $\Lambda_{\rm QCD}$, as a result of nonperturbative quantum chromodynamics (QCD) [8,9]. Physically, this is a satisfying result, because we believe that quarks are permanently confined within hadrons, precluding the unambiguous definition of a quark pole mass [10].

The top quark decays very quickly, having a width $\Gamma \approx 1.5$ GeV, approximately an order of magnitude greater than the strong-interaction energy scale $\Lambda_{\rm QCD} \approx 200$ MeV. Such a short lifetime means that the top quark decays before it has time to hadronize [11–13]. The large top-quark width can act as an infrared cutoff in the calculation of physical quantities, insulating the top quark from the effects of nonperturbative QCD [12,14–16].

Motivated by these facts, we ask whether the top-quark pole mass is free of the ambiguities associated with nonperturbative QCD. The purpose of this article is to demonstrate that this is not the case. The top-quark pole mass, like the pole mass of a stable heavy quark, is unavoidably ambiguous by an amount proportional to $\Lambda_{\rm QCD}$. We first give a general argument for the existence of such an ambiguity. We then give a heuristic argument that the ambiguity is proportional to $\Lambda_{\rm QCD}$, using the specific example of the Wb invariant-mass distribution. Finally, we prove that the ambiguity is proportional to $\Lambda_{\rm QCD}$ by using infrared renormalons.

Consider a scattering process with asymptotic states consisting of stable particles. We ask if it is possible for the scattering amplitude to have a pole at the mass of a stable quark. This would correspond to a quark propagator connecting two subamplitudes, as depicted in Fig. 1; the pole in the quark propagator would correspond to the pole in the amplitude. Such a configuration is impossible, however, because the subamplitudes which the quark propagator connects have external states which are color singlets (due to confinement), while the quark is a color triplet, so color is not conserved. Thus there cannot be a pole in the amplitude at the quark mass.

This argument applies equally well to an unstable quark, such as the top quark. The fact that the quark is unstable evidently plays no role in the argument; it only shifts the imagined pole in the propagator into the complex plane. As in the case of a stable quark, there cannot be a pole in the amplitude, regardless of how short

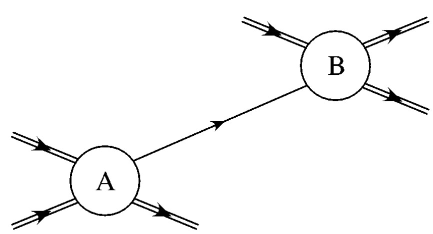

FIG. 1. A scattering amplitude consisting of two subamplitudes connected by a quark propagator. The external lines represent color-singlet asymptotic states. Such an amplitude is forbidden by color conservation.

lived the quark. In particular, the fact that the top-quark lifetime is much less than $\Lambda_{\rm QCD}^{-1}$ is irrelevant.

Such an argument implies that the nonperturbative aspect of the strong interaction will stand in the way of any attempt to unambiguously extract the top-quark pole mass from experiment. For example, consider the extraction of the pole mass from the peak in the Wb invariant-mass distribution. In perturbation theory, the final state is a W and a b quark, as depicted in Fig. 2(a). However, the b quark manifests itself experimentally as a jet of colorless hadrons, due to confinement. At least one of the quarks which resides in these hadrons comes from elsewhere in the diagram, and cannot be considered as a decay product of the top quark, as depicted in Fig. 2(b). This leads to an irreducible uncertainty in the Wb invariant mass of $O(\Lambda_{\rm QCD})$ and, hence, an ambiguity of this amount in the extracted top-quark pole mass.

We now turn to an investigation of the top-quark pole mass from the perspective of *infrared renormalons*. We first review the argument which demonstrates the existence of a renormalon ambiguity in the pole mass of a stable heavy quark [8,9]. We then extend the argument to take into account the finite width of the top quark. Finally, we investigate the existence of a renormalon ambiguity in the top-quark width itself.

The pole mass of a quark is defined by the position of the pole in the quark propagator. The propagator of a quark of four-momentum p is

$$D(\not p) = \frac{i}{\not p - m_R - \Sigma(\not p)}, \qquad (1)$$

where m_R is a renormalized short-distance mass [by short-distance mass we mean a running mass (such as the $\overline{\rm MS}$ mass) evaluated at a scale $\mu \gg \Lambda_{\rm QCD}$], and $\Sigma(\not p)$ is the renormalized one-particle irreducible quark self-energy. The equation for the position of the pole is an implicit equation that can be solved perturbatively:

$$\not p_{\rm pole} = m_R + \Sigma(\not p_{\rm pole}) = m_R + \Sigma^{(1)}(m_R) + \ldots, \qquad (2)$$

where $\Sigma^{(1)}(m_R)$ is the one-loop quark self-energy shown in Fig. 3(a). This quantity is real, so the pole position is real.

Renormalons arise from the class of diagrams generated by the insertion of n vacuum-polarization subdiagrams into the gluon propagator in the one-loop self-energy diagram, as shown in Fig. 3(a′). One can express this as

$$\Sigma^{(1)}(m_R, a) = \frac{16 m_R}{3\beta_0} \sum_{n=0}^{\infty} c_n a^{n+1}, \qquad (3)$$

where

$$a \equiv \frac{\beta_0 \alpha_s(m_R)}{4\pi} \qquad (4)$$

and β_0 is the one-loop QCD beta-function coefficient, $\beta_0 \equiv 11 - (2/3)N_f$. Formally, these are the dominant QCD corrections in the "large-β_0" limit. Thus $\Sigma^{(1)}(m_R, a)$ in Eq. (3) is calculated at leading order in α_s, but to all orders in a.

For large n the coefficients c_n grow factorially, and are given by [8,9,17]

$$c_n \overset{n\to\infty}{\to} e^{-C/2} 2^n n!, \qquad (5)$$

where C is a finite renormalization-scheme-dependent constant (in the $\overline{\rm MS}$ scheme, $C = -5/3$). The series in Eq. (3) is therefore divergent. One can attempt to sum the series using the technique of Borel resummation [18]. The Borel transform (with respect to a) of the self-energy is obtained from the series coefficients, Eq. (5), via

$$\tilde\Sigma^{(1)}(m_R, u) = \frac{16 m_R}{3\beta_0} \sum_{n=0}^{\infty} \frac{c_n}{n!} u^n, \qquad (6)$$

where u is the Borel parameter. Because the coefficients c_n are divided by $n!$ in the above expression, the series has a finite radius of convergence in u, and can be analytically continued into the entire u plane. The self-energy is then reconstructed via the inverse Borel transform, given formally by

$$\Sigma^{(1)}(m_R, a) = \int_0^\infty du\, e^{-u/a} \tilde\Sigma^{(1)}(m_R, u). \qquad (7)$$

The integral in Eq. (7) is only formal, because the Borel transform of the quark self-energy possesses singularities on the real-u axis, which impede the evaluation of the integral. These singularities are referred to as infrared renormalons because they arise from the region of soft gluon momentum in Fig. 3(a′). The series for the self-energy in Eq. (3) is therefore not Borel summable.

The divergence of the series for the self-energy is governed by the infrared renormalon closest to the origin, which lies at $u = 1/2$. This renormalon is not associated

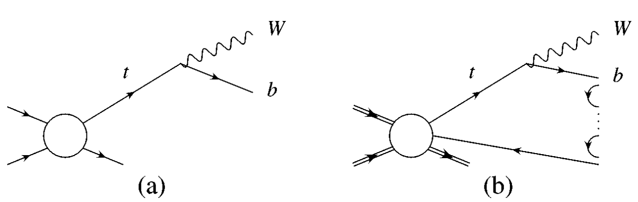

FIG. 2. The production and decay of a top quark in (a) perturbation theory and (b) nonperturbatively.

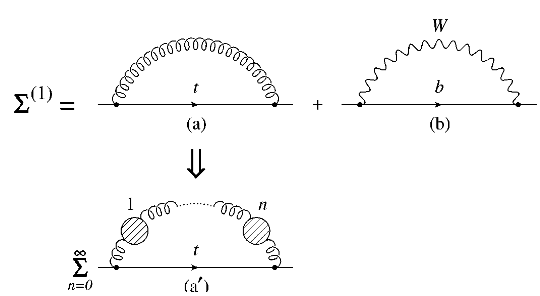

FIG. 3. Diagrams contributing to the top quark self-energy at leading order in α_s and α_W. (a′) replaces (a) when summing to all orders in $\beta_0 \alpha_s$.

with the condensate of a local operator, so it cannot be absorbed into a nonperturbative redefinition of the pole mass [8,9]. Instead, one can choose some *ad hoc* prescription to circumvent the singularity in the integral. The difference between various prescriptions is a measure of the ambiguity in the pole mass. Estimating the ambiguity as half the difference between deforming the integration contour above and below the singularity gives [9]

$$\delta m_{\text{pole}} \sim \frac{8\pi}{3\beta_0} e^{-C/2} \Lambda_{\text{QCD}}, \tag{8}$$

so the pole mass is ambiguous by an amount proportional to Λ_{QCD}.

We now include the $O(\alpha_W)$ contribution to the top-quark self-energy shown in Fig. 3(b). The pole position is still given by Eq. (2), but where $\Sigma^{(1)}(m_R)$ includes both Figs. 3(a) and 3(b). Since Fig. 3(b) has an imaginary part, the pole moves off the real axis. The imaginary part of the one-loop pole position defines the tree-level top-quark width via $\text{Im}\,\Sigma^{(1)}(m_R) \equiv -\frac{1}{2}\Gamma_{\text{tree}}$. As before, to extend the calculation to all orders in a, we replace Fig. 3(a) by Fig. 3(a′). This contribution to the pole mass remains the same as for a stable quark, and has the same renormalon ambiguity. Thus, at leading order in α_W, the infrared renormalons do not know about the top-quark width.

The $O(\alpha_s)$ contribution to the top-quark self-energy learns about the top-quark width if one works to all orders in α_W, via a Schwinger-Dyson representation [19], as shown in Fig. 4. The circles on the internal propagators and the vertex in Figs. 4(a) and 4(b) represent the weak corrections to all orders in α_W [the circles in Fig. 4(b) also contain one power of α_s]. We wish to solve for the pole position as given by the first equality in Eq. (2). We denote the pole position at zeroth order in α_s, but to all orders in α_W, by the complex value M, with $\text{Im}\,M \equiv -\frac{1}{2}\Gamma$, where Γ is the top-quark width to all orders in α_W. At leading order in α_s, the pole position is then given by

$$\not{p}_{\text{pole}} = m_R + \Sigma(M), \tag{9}$$

where $\Sigma(M)$ is given by Figs. 4(a) and 4(b). Again, we extend this calculation to all orders in a by making n vacuum-polarization insertions in the gluon propagator,

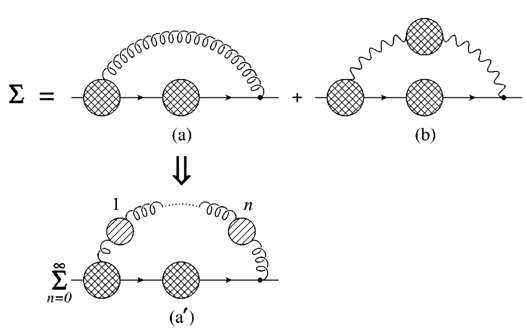

FIG. 4. Diagrams contributing to the top quark self-energy at leading order in α_s, but to all orders in α_W. (a′) replaces (a) when summing to all orders in $\beta_0\alpha_s$.

as depicted in Fig. 4(a′). This yields a series in a, which we denote by $\Sigma(M,a)$ in analogy with Eq. (3). To investigate whether the width might cut off the infrared renormalons generated by these diagrams, we need only consider the contribution of soft gluons. In the limit of vanishing gluon momentum, the internal quark propagator reduces to $Z/(\not{p} - M)$, where Z is the wave-function renormalization factor. The Ward identity tells us that, in this same limit, the dressed vertex is simply Z^{-1}. Thus, in the infrared limit, $\Sigma(M,a)$ is formally identical to $\Sigma^{(1)}(m_R,a)$ with m_R replaced by M everywhere. The infrared renormalons, which are associated with the Borel transform with respect to a, are unaffected. The width does not act as a cutoff for infrared renormalons, despite the fact that it is much greater than Λ_{QCD}. We conclude that the ambiguity in the pole mass of the top quark is given by Eq. (8), just as for a stable quark.

We now ask whether the top-quark width suffers from a similar renormalon ambiguity. Because the first-order calculation yields the top-quark width at tree level only, it is insufficient to address this question. The solution to Eq. (2) at $O(\alpha_W\alpha_s)$ is

$$\begin{aligned} \not{p}_{\text{pole}} &= m_R + \Sigma(m_R + \Sigma(m_R)) \\ &= m_R + \Sigma^{(1)}(m_R) + \Sigma^{(2)}(m_R) \\ &\quad + \Sigma^{(1)\prime}(m_R)\Sigma^{(1)}(m_R), \end{aligned} \tag{10}$$

where the superscripts on Σ indicate the order at which it is to be evaluated. The imaginary part of this equation (times $-1/2$) defines the top-quark width at $O(\alpha_W\alpha_s)$.

One may calculate the imaginary part of Eq. (10) using the Cutkosky rules. This reduces to the calculation of the QCD correction to the process $t \to Wb$ [the term involving $\Sigma^{(1)\prime}(m_R)$ corresponds to the wave-function renormalization of the top quark]. The presence of renormalons in this process was investigated in Refs. [8,20]. If the width is expressed in terms of the pole mass, then it has an infrared renormalon at $u = 1/2$, corresponding to an ambiguity proportional to Λ_{QCD}. However, if the width is expressed in terms of a short-distance mass, such as the $\overline{\text{MS}}$ mass, there is no renormalon at $u = 1/2$, and hence no ambiguity proportional to Λ_{QCD}.

Let us summarize our results. The confinement of color, a nonperturbative property of QCD, precludes the existence of S-matrix poles at quark masses and impedes any attempt to unambiguously define the pole mass of a stable heavy quark. The same is true of the top-quark pole mass despite the fact that the top-quark width is much greater than the strong-interaction energy scale, Λ_{QCD}. This is signaled by the divergent behavior at large orders of the expansion of the top-quark self-energy in powers of $a = \beta_0\alpha_s(m_R)/4\pi$, which leads to an unavoidable ambiguity of $O(\Lambda_{\text{QCD}})$ in the top-quark pole mass. The top-quark pole mass is therefore not a physical quantity. The top-quark width does not suffer from an ambiguity of the same order.

The ambiguity in the pole mass does not limit the accuracy with which a short-distance mass, such as the $\overline{\rm MS}$ mass, can be measured. It is sensible to adopt the $\overline{\rm MS}$ mass as the standard definition of the top-quark mass, as is the convention for the lighter quarks [21]. The relation between the top-quark pole mass and the $\overline{\rm MS}$ mass evaluated at the pole mass, $\overline{m}(m_{\rm pole})$, is known to two loops [22] ($\overline{\alpha}_s(\mu)$ is the $\overline{ms}$ coupling evaluated at the scale μ):

$$m_{\rm pole} = \overline{m}(m_{\rm pole})\left[1 + \frac{4}{3}\frac{\overline{\alpha}_s(m_{\rm pole})}{\pi} + 10.95\left(\frac{\overline{\alpha}_s(m_{\rm pole})}{\pi}\right)^2 + \ldots\right] + O(\Lambda_{\rm QCD}), \tag{11}$$

where the last term reminds us that the pole mass has an unavoidable ambiguity of $O(\Lambda_{\rm QCD})$. Given that the pole mass is ambiguous, we suggest as the standard the $\overline{\rm MS}$ mass evaluated at the $\overline{\rm MS}$ mass, which is related to the pole mass by

$$m_{\rm pole} = \overline{m}(\overline{m})\left[1 + \frac{4}{3}\frac{\overline{\alpha}_s(\overline{m})}{\pi} + 8.28\left(\frac{\overline{\alpha}_s(\overline{m})}{\pi}\right)^2 + \ldots\right] + O(\Lambda_{\rm QCD}). \tag{12}$$

The difference in the coefficients of the two $\overline{\alpha}_s^2$ terms above is 8/3. For a top-quark pole mass of 175.6 ± 5.5 GeV [2], $\overline{m}(\overline{m}) = 166.5 \pm 5.5$ GeV [$\overline{m}(m_{\rm pole}) = 166.0 \pm 5.5$ GeV].

The considerations of this paper apply to any colored particle, stable or unstable. Thus, if nature is supersymmetric, the pole masses of squarks and gluinos will necessarily be ambiguous by an amount proportional to $\Lambda_{\rm QCD}$.

We are grateful for conversations with E. Braaten, A. El-Khadra, R. Leigh, T. Liss, and T. Stelzer. This work was supported in part by Department of Energy Grant No. DE-FG02-91ER40677. We gratefully acknowledge the support of GAANN, under Grant No. DE-P200A10532, from the U.S. Department of Education for M.S.

[1] CDF Collaboration, F. Abe *et al.*, Phys. Rev. Lett. **74**, 2626 (1995); D0 Collaboration, S. Abachi *et al.*, Phys. Rev. Lett. **74**, 2632 (1995).

[2] A. Yagil, International Europhysics Conference on High Energy Physics, Jerusalem, 1997 (to be published).

[3] Future Electroweak Physics at the Fermilab Tevatron: Report of the tev_2000 Study Group, edited by D. Amidei and R. Brock, Report No. FERMILAB-Pub-96/082 (1996).

[4] ATLAS Technical Proposal, 1994, Report No. CERN/LHCC/94-43, LHCC/P2.

[5] NLC ZDR Design Group and the NLC Physics Working Group, Report No. hep-ex/9605011.

[6] $\mu^+\mu^-$ Collider Collaboration, 1996, Report No. BNL-52503.

[7] R. Tarrach, Nucl. Phys. **B183**, 384 (1981).

[8] I. Bigi, M. Shifman, N. Uraltsev, and A. Vainshtein, Phys. Rev. D **50**, 2234 (1994).

[9] M. Beneke and V. Braun, Nucl. Phys. **B426**, 301 (1994); M. Beneke, Phys. Lett. B **344**, 341 (1995); P. Ball, M. Beneke, and V. Braun, Nucl. Phys. **B452**, 563 (1995).

[10] I. Bigi and N. Uraltsev, Phys. Lett. B **321**, 412 (1994).

[11] J. Kühn, Acta Phys. Austriaca Suppl. **XXIV**, 203 (1982).

[12] I. Bigi, Y. Dokshitzer, V. Khoze, J. Kühn, and P. Zerwas, Phys. Lett. B **181**, 157 (1986).

[13] L. Orr and J. Rosner, Phys. Lett. B **246**, 221 (1990); L. Orr, Phys. Rev. D **44**, 88 (1991).

[14] V. Fadin and V. Khoze, Pis'ma Zh. Eksp. Teor. Fiz. **46**, 417 (1987) [JETP Lett. **46**, 525 (1987)]; Yad. Fiz. **48**, 487 (1988) [Sov. J. Nucl. Phys. **48**, 309 (1988)]; V. Fadin, V. Khoze, and T. Sjöstrand, Z. Phys. C **48**, 613 (1990).

[15] M. Strassler and M. Peskin, Phys. Rev. D **43**, 1500 (1995).

[16] V. Khoze and T. Sjöstrand, Phys. Lett. B **328**, 466 (1994).

[17] K. Philippides and A. Sirlin, Nucl. Phys. **B450**, 3 (1995).

[18] G. 't Hooft, in *The Whys of Subnuclear Physics, Proceedings of the International School of Subnuclear Physics, Erice, 1977*, edited by A. Zichichi (Plenum, New York, 1979), p. 943; A. Mueller, in *QCD—20 Years Later, Proceedings of the Workshop, Aachen, Germany, 1992*, edited by P. Zerwas and H. Kastrup (World Scientific, Singapore, 1993), Vol. 1, p. 162.

[19] C. Itzykson and J.B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980), p. 475.

[20] M. Beneke, V. Braun, and V. Zakharov, Phys. Rev. Lett. **73**, 3058 (1994); M. Beneke and V. Braun, Phys. Lett. B **348**, 513 (1995).

[21] *Review of Particle Properties*, Phys. Rev. D **54**, 1 (1996) (see p. 303).

[22] N. Gray, D. Broadhurst, W. Grafe, and K. Schilcher, Z. Phys. C **48**, 673 (1990).