Supersymmetry between Phase-Equivalent Coupled-Channel Potentials

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With supersymmetric quantum mechanics, a coupled-channel potential is determined which is phase equivalent to a given one, and whose bound spectrum is identical except for one arbitrary bound state which is removed. The bound state is suppressed by a first supersymmetric transformation and the original scattering matrix is recovered with a second transformation. The resulting potential presents an r^{-2} singularity at the origin in some channels. The method is applied to the removal of the nonphysical state of the deep $^3S_1$-$^3D_1$ neutron-proton Moscow potential and transforms it into a shallow potential with a repulsive core. [S0031-9007(97)04528-6]

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A simple way for describing the interaction between two composite particles is to use a local potential. Such a potential can often reproduce both the bound states formed by the interacting particles, and their scattering properties. However, since these particles have an internal structure, an ambiguity may arise between different potential families: *shallow* potentials, which possess physical bound states only, and *deep* potentials, which in addition have nonphysical bound states simulating the effect of the Pauli principle between the constituent fermions [1]. Despite such differences, potentials may be phase equivalent; i.e., they may share the same scattering matrix at all energies. Studying the relations between phase-equivalent potentials differing by their bound spectrum is thus an important physical problem.

In the single-channel case, supersymmetric quantum mechanics [2] provides a powerful tool for performing this study, since it allows removing bound states from a given deep potential without modifying its phase shift [3]. More general transformations (addition of bound states, modification of a bound-state energy) are also possible [4–6], and provide the most general form of phase-equivalent potentials for arbitrary modifications of the bound spectrum [7].

An ambiguity between deep and shallow potentials also appears in coupled-channel cases. An important example in nuclear physics is the coexistence of very different families of nucleon-nucleon potentials: shallow potentials [8–10], which display only the deuteron bound state at -2.22 MeV in the $^3S_1$-$^3D_1$ channel, and deep potentials [11–13], which have an additional nonphysical bound state simulating the underlying quark structure (see Ref. [14] for details). To analyze such ambiguities, the derivation of phase-equivalent potentials must be extended to coupled channels, and supersymmetric quantum mechanics is an ideal tool to perform it. In Refs. [15,16], the supersymmetric formalism is generalized to the coupled-channel case. However, an attempt to perform a phase-equivalent transformation removing a bound state failed [17]. The aim of this Letter is to show that a phase-equivalent bound-state removal is in fact possible, provided sufficiently general supersymmetric transformations are used.

In this Letter, we consider for a given partial wave a Hamiltonian H_0 coupling n two-body channels with equal masses m for simplicity, but with arbitrary thresholds. The system of n coupled Schrödinger equations at energy E (in units $\hbar = 2m = 1$) reads

$$[-d^2/dr^2 + V_0(r)]\Phi_0(E,r) = E\Phi_0(E,r)\,. \qquad (1)$$

The effective potential V_0 is an $n \times n$ Hermitian matrix, which tends to

$$V_0(r) \xrightarrow[r\to\infty]{} \Delta \qquad (2)$$

asymptotically. The $n \times n$ matrix Δ is real and diagonal, and its diagonal element Δ^{ii} is the threshold of channel i ($i = 1, \ldots, n$). A solution Φ_0 of the system can be either a column eigenvector or a matrix whose columns are such eigenvectors. The "0" subscripts distinguish the initial Hamiltonian H_0 and its solutions from others that are built in the following.

Let us assume that this system has at least one bound state at real energy $\mathcal{E}$ lower than all thresholds; i.e., a normalized eigenvector ψ_0^1 exists at this energy [$\int_0^\infty \psi_0^{1\dagger}(\mathcal{E},t)\psi_0^1(\mathcal{E},t)\,dt = 1$, where $\psi^\dagger$ is the adjoint row vector of ψ]. In this Letter, we construct a new potential which has the same S matrix and the same bound spectrum as V_0, with the exception of $\mathcal{E}$ which is removed. It will be proved below that the Hamiltonian H_2 with potential matrix

$$V_2(r) = V_0(r) - 2\frac{d}{dr}\frac{\psi_0^1(\mathcal{E},r)\psi_0^{1\dagger}(\mathcal{E},r)}{\int_0^r \psi_0^{1\dagger}(\mathcal{E},t)\psi_0^1(\mathcal{E},t)\,dt} \qquad (3)$$

satisfies these properties. The subscript "2" refers to the number of supersymmetric transformations applied to the Hamiltonian. Equation (3) is valid with or without thresholds. The removed state may be any bound state of the system, as in the single-channel case [6], since the denominator is always positive. All solutions $\Phi_2(E,r)$

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of H_2 that are bounded at infinity, and in particular its bound and scattering states, can be expressed in terms of solutions Φ_0 of H_0 as

$$\Phi_2(E) = \Phi_0(E) + \psi_0^1(\mathcal{E}) \frac{\int_r^\infty \psi_0^{1\dagger}(\mathcal{E})\Phi_0(E)\,dt}{\int_0^r \psi_0^{1\dagger}(\mathcal{E})\psi_0^1(\mathcal{E})\,dt}, \quad (4)$$

where Φ_0 and Φ_2 can be either vectors or matrices (we sometimes omit the position variables for brevity). This equation shows that Φ_0 and Φ_2 have the same asymptotic behavior, which means that V_0 and V_2 are phase equivalent. When $\Phi_0 = \psi_0^1$, a study of Eq. (4) at small r shows that the initial bound state does not transform into a square-integrable state. However, proving that H_2 has no bound state at this energy is more delicate and will be discussed below.

Equations (3) and (4) can also be proved by direct verification. They can be used numerically without major difficulty: Only bound- and scattering-state calculations for the initial Schrödinger equation are needed, followed by simple integrations and derivations. The reader mainly interested in the physical content of the method may directly proceed to the neutron-proton example.

We now establish these results with two supersymmetric transformations. The initial coupled-channel Hamiltonian is factorized as [15]

$$H_0 \equiv -d^2/dr^2 + V_0(r) = A_0^+ A_0^- + \mathcal{E}, \quad (5)$$

where $\mathcal{E}$ is the factorization energy, and $A_0^\pm$ read

$$A_0^\pm = (A_0^\mp)^\dagger = \pm d/dr + U_0(r), \quad (6)$$

with the $n \times n$ superpotential matrix

$$U_0(r) = \Psi_0'(\mathcal{E}, r)\Psi_0^{-1}(\mathcal{E}, r). \quad (7)$$

In Eq. (7), the factorization solution Ψ_0 is a matrix, solution of (1) at energy $\mathcal{E}$, and the prime means a derivation with respect to r. Notice that U_0 is defined only if the columns of Ψ_0 are linearly independent.

The partner Hamiltonian

$$H_1 = A_0^- A_0^+ + \mathcal{E} \quad (8)$$

corresponds to a new effective potential

$$V_1(r) = V_0(r) - 2U_0'(r). \quad (9)$$

This potential is Hermitian when Ψ_0 is self-conjugate [16], i.e., when [18]

$$W[\Psi_0(\mathcal{E}, r), \Psi_0(\mathcal{E}, r)] = 0 \quad (10)$$

for any r, where the Wronskian is defined as $W[\Psi, \Phi] \equiv \Psi^\dagger \Phi' - \Psi'^\dagger \Phi$.

This supersymmetric transformation removes the bound state from the spectrum of the initial Hamiltonian H_0 if Ψ_0 reads

$$\Psi_0(\mathcal{E}, r) = [\psi_0^1(\mathcal{E}, r), \psi_0^2(\mathcal{E}, r), \ldots, \psi_0^n(\mathcal{E}, r)], \quad (11)$$

where ψ_0^1 is the bound-state eigenvector, and the ψ_0^i for $i = 2, \ldots, n$ are linearly independent column eigenvectors of H_0 with eigenvalue $\mathcal{E}$. In Refs. [15,16], these other eigenvectors are chosen regular at the origin; hence, they exponentially grow at infinity. As shown in Ref. [17], this choice does not allow the construction of phase-equivalent potentials. We make a different choice here: The other eigenvectors are chosen to exponentially decrease at infinity, but are then singular at the origin. Thus, for a diagonal matrix κ with $\kappa^{ii} = (-\mathcal{E} + \Delta^{ii})^{1/2}$, one has

$$\Psi_0(\mathcal{E}, r) \underset{r\to\infty}{\sim} \exp(-\kappa r) C_0, \quad (12)$$

where C_0 is an invertible constant matrix, so that the columns of Ψ_0 are linearly independent. The thresholds are not affected by the transformation [see Eqs. (7) and (9)]. Equation (12) shows that $W[\Psi_0, \Psi_0]$ vanishes at infinity. When a Wronskian of solutions of Eq. (1) is zero at infinity, it can be written as

$$W[\Phi(E, r), \Psi(F, r)] = (F - E) \int_r^\infty \Phi^\dagger(E)\Psi(F)\,dt, \quad (13)$$

which implies Eq. (10).

For $E \neq \mathcal{E}$, the solutions of H_1 can be deduced from the intertwinning relation $A_0^- H_0 = H_1 A_0^-$ and read

$$\Phi_1(E, r) = A_0^- \Phi_0(E, r) \quad (14a)$$

$$= \Psi_0^{-1\dagger}(\mathcal{E}, r) W[\Phi_0(E, r), \Psi_0(\mathcal{E}, r)]^\dagger \quad (14b)$$

$$= (\mathcal{E} - E)\Psi_0^{-1\dagger}(\mathcal{E}) \int_r^\infty \Psi_0^\dagger(\mathcal{E})\Phi_0(E)\,dt. \quad (14c)$$

The asymptotic behavior of Eq. (14a) for positive E (above all thresholds) provides the modification of the initial S matrix S_0 into [15,16]

$$S_1(k) = (-ik - \kappa) S_0(k) (ik - \kappa)^{-1}, \quad (15)$$

where k is a diagonal matrix with elements $k^{ii} = (E - \Delta^{ii})^{1/2}$. The pole at $k = i\kappa$ is suppressed, and hence H_1 has no bound state at $\mathcal{E}$. Moreover, since $S_1(k) \neq S_0(k)$, V_1 is not phase equivalent to V_0.

A second factorization leading to a new Hamiltonian H_2 (i.e., a factorization of H_1 with Eqs. (5)–(7) where subscript "0" is replaced by subscript "1") restores the S matrix when [17]

$$S_2(k) = (-ik + \kappa) S_1(k) (ik + \kappa)^{-1}. \quad (16)$$

Indeed, since $(k^{ii})^2 + (\kappa^{ii})^2 = E - \mathcal{E}$ for all channels, Eqs. (15) and (16) imply phase equivalence

$$S_2(k) = S_0(k)\,. \tag{17}$$

Equation (16) is true for a self-conjugate factorization matrix Ψ_1, solution of H_1 at energy $\mathcal{E}$ and satisfying

$$\Psi_1(\mathcal{E}, r) \underset{r\to\infty}{\sim} \exp(\kappa r) C_1\,, \tag{18}$$

where C_1 is a constant invertible matrix. A direct verification shows that $\Psi_0^{-1\dagger}$ is such a solution, with $C_1 = C_0^{-1\dagger}$. For constant matrices K and L with $L^\dagger K = K^\dagger L$, other self-conjugate solutions read [18]

$$\Psi_1(\mathcal{E}, r) = \Psi_0^{-1\dagger}(\mathcal{E}, r) \times \left\{K - \left[\int_r^\infty \Psi_0^\dagger(\mathcal{E}, t)\Psi_0(\mathcal{E}, t)\, dt\right] L\right\}, \tag{19}$$

and satisfy (18) with K invertible.

Phase equivalence (17) implies that S_2 has a pole at $i\kappa$ since H_0 has a bound state at energy $\mathcal{E}$. However, as in the single-channel case [19], this pole may be due to a pole of the Jost matrix in the lower half k plane, and not to a zero of the Jost matrix in the upper half k plane as for a bound state [20]. Below, we show in a particular case that the second transformation does not reintroduce a bound state at energy $\mathcal{E}$ when K and L are chosen as

$$K^{ij} = \delta_{ij}, \qquad L^{ij} = \delta_{i1}\delta_{j1}\,. \tag{20}$$

Using Eqs. (19) and (20) to construct $U_1 = \Psi_1'\Psi_1^{-1}$ and $V_2 = V_1 - 2U_1'$ leads to Eq. (3). The thresholds are not affected by the second transformation. For $E \neq \mathcal{E}$, the solutions of H_1 are expressed in terms of those of H_0 with Eq. (14c) and the solutions of H_2 in terms of those of H_1 with Eq. (14a) (with subscripts increased by 1). Combining these two results leads to Eq. (4).

At the factorization energy, $\Psi_1^{-1\dagger}$ is solution of H_2. It consists of n eigenvectors exponentially decreasing and linearly independent at infinity, as deduced from (18). The new Hamiltonian H_2 has thus no bound state at $\mathcal{E}$ if these vectors are singular and linearly independent at the origin, i.e., if the columns of Ψ_1 are regular and linearly independent at the origin. This behavior can be studied with a series expansion of Eq. (19), which provides conditions on K and L. Such a study cannot be performed in all generality in this Letter because it requires assumptions on the behavior of the potential at the origin. We limit ourselves to a particular case which simplifies calculations. However, Eqs. (3) and (4) are valid in a more general context.

For simplicity, the initial potential V_0 is assumed here to behave near the origin like

$$V_0(r) \underset{r\to 0}{\sim} \nu_0(\nu_0 + 1) r^{-2}, \tag{21}$$

where ν_0 is a real and diagonal matrix. Moreover, we assume that the potential of channel 1 is less singular than the others: $\nu_0^{11} < \nu_0^{ii}$ for $i = 2, \ldots, n$. This is the case in the example treated below. With these assumptions, both V_1 and V_2 behave like V_0 in Eq. (21) but with different ν and the corresponding systems of equations decouple near the origin. For example, the leading terms of Ψ_0 read

$$\Psi_0(\mathcal{E}, r) \underset{r\to 0}{\sim} (r^{\nu_0+1} b_0^1, r^{-\nu_0} b_0^2, \ldots, r^{-\nu_0} b_0^n)\,, \tag{22}$$

where the b_0^i are constant column vectors. The first column of Ψ_0 is regular at the origin, while the $n - 1$ others are singular. In the same way, imposing the regularity of Ψ_1 leads to Eqs. (20).

Equations (7), (9), and (22) provide the behavior of V_1 at the origin: $\nu_1^{11} = \nu_0^{11} + 1$ and $\nu_1^{ii} = \nu_0^{ii} - 1$ $(i = 2, \ldots, n)$. The behavior of V_2 at the origin can be deduced from that of Ψ_1 and reads like in Eq. (21) with $\nu_2^{ii} = \nu_1^{ii} + 1$ for all i. Combining these results allows expressing the V_2 singularity in terms of the V_0 singularity by

$$\nu_2^{11} = \nu_0^{11} + 2\,, \tag{23a}$$

$$\nu_2^{ii} = \nu_0^{ii} \qquad (i = 2, \ldots, n)\,. \tag{23b}$$

The lowest singularity parameter increases by two units (as in the single-channel case [3]), while all others remain unchanged. When one diagonal potential of V_0 is not assumed to be less singular than the others, a more complicated result is obtained: The increase of the singularity is shared between the least singular channels, and their coupling potentials also become singular.

We now apply the method to a neutron-proton $^3S_1$-$^3D_1$ potential. We start from the D-type Moscow potential of Ref. [13], which has a nonphysical bound state at -561 MeV, simulating a Pauli forbidden state. The

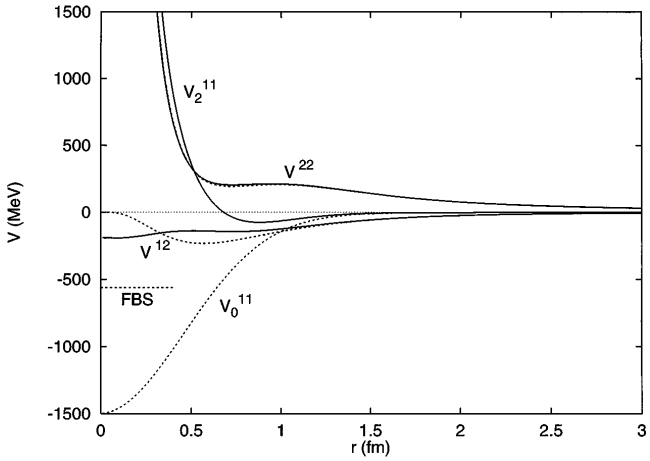

FIG. 1. Neutron-proton $^3S_1$-$^3D_1$ Moscow potential (V_0, dashed lines) of Ref. [13] with a Pauli forbidden bound state (FBS) at -561 MeV, and phase-equivalent shallow potential (V_2, solid lines) obtained by removal of this bound state with Eq. (3). The effective potentials are represented with superscripts referring to the channel as in the text (1 for S wave, 2 for D wave).

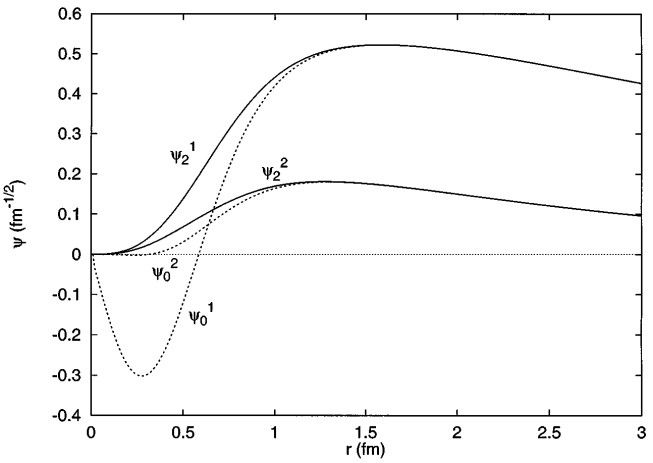

FIG. 2. Deuteron normalized wave function of the deep potential V_0 (ψ_0, dashed lines), and of its supersymmetric shallow partner V_2 (ψ_2, solid lines), as obtained by Eq. (4) (superscript 1 for S wave, 2 for D wave).

effective potentials [Eqs. (33)–(37) of Ref. [13]] are represented in Fig. 1 by dashed lines. There is no threshold ($\Delta^{11} = \Delta^{22} = 0$), and one has $\nu_0^{11} = 0$ (S wave) and $\nu_0^{22} = 2$ (D wave). By removing the deep bound state, a shallow effective potential is obtained, represented by solid lines in Fig. 1. The singularity of the S-wave potential increases by two following Eq. (23a), and creates a repulsive core in qualitative agreement with existing shallow potentials [8–10]. A similar result was obtained in Ref. [14] with an approximate method for a previous version of the Moscow potential [11]. Strikingly, the D-wave potential and the coupling term are not much modified.

Physical properties of both potential families are compared in Refs. [11–14]. Let us briefly discuss here the deuteron wave function. Both the initial and transformed potentials have a physical bound state at -2.22 MeV, corresponding to the deuteron ground state. The S and D components of this bound state are represented in Fig. 2. The wave function of the deep potential, obtained by numerical resolution of Eq. (1), has a node in both S and D channels, but only the S-wave node is significant (in previous versions of the Moscow potential [11], the D wave also had a marked node, whose existence does not agree with microscopic quark models [13]). These nodes disappear in the wave function of the shallow potential, calculated by Eq. (4), and are replaced by an r^3 behavior near the origin, in agreement with the singularity modification (23). Figure 2 also shows that both wave functions have very close asymptotic behaviors. This is also true for scattering states, which confirms the phase equivalence between both potentials.

In conclusion, supersymmetric quantum mechanics allows the construction of phase-equivalent potentials in the coupled-channel case, as it does in the single-channel case. Previous attempts to derive phase-equivalent potentials by supersymmetry were unsuccessful because of a too restricted choice of factorization solutions. The central result of this Letter [Eqs. (3) and (4)] is valid under broader circumstances than the case for which the proof is discussed. A systematic study of all supersymmetric transformations in the coupled-channel case should lead, as in the single-channel case, (i) to arbitrary modifications of the bound spectrum without modification of the S matrix [7], and (ii) to approximate solutions of the inverse problem with rational S-matrix expansions [21]. This general study is under way.

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