Superconducting Quantum Critical Point

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We study the properties of a quantum critical point which develops in a BCS superconductor when pair breaking suppresses the transition temperature to zero. The pair fluctuations are characterized by a dynamical critical exponent $z = 2$. Except for very low temperatures, anomalous contribution to the conductivity is proportional to $\sqrt{T}$ in three dimensions, but to $1/T$ in two dimensions. At lowest temperatures, the conductivity correction varies as $T^{1/4}$ in three dimensions, and as $\ln(1/T)$ in two. [S0031-9007(97)04458-X]

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The possibility of quantum critical behavior in itinerant magnets has attracted great attention in recent years, in part, because quantum criticality affords the possibility of a controlled study of non-Fermi-liquid behavior [1]. At a quantum critical point, order parameter fluctuations develop an infinite correlation range in both space *and* time [2,3]. The coupling between these fluctuations and conducting electron fluid is able, under certain circumstances, to eliminate the formation of well-defined quasiparticles in the electron liquid, giving rise to a new kind of metallic behavior.

In this paper we discuss the possibility of quantum critical behavior in superconductors. A quantum critical point implies a finite value of the electron interaction strengths. At first sight, this would appear to rule out the possibility of a superconducting quantum critical point, for conventional superconductivity develops for an arbitrarily small pair interaction. If the transition temperature is driven to zero in a pure BCS superconductor, the pairing interaction and the pair fluctuations are completely eliminated. Fortunately, this is not the case in the presence of pair breaking, which cuts off the logarithmic singularity in the pair susceptibility, requiring that the pairing interaction reach a critical strength before superconductivity develops. In this paper we characterize the quantum critical behavior which develops at this special point. Our results can be tested experimentally on conventional superconductors, such as Ce-doped La [4]. They may also provide a useful diagnostic tool for the understanding of unconventional, e.g., heavy fermion, superconductors.

We begin with writing down the effective action of the pair field Δ in the vicinity of a quantum critical point:

$$S_{\text{eff}}[\Delta] = T \sum_{\nu,k} \Delta^{\dagger} \chi^{-1} \Delta + \frac{N(0)T^3}{T_0^2} \sum_{1,2,3} \Delta_1^{\dagger} \Delta_2^{\dagger} \Delta_3 \Delta_4 , \tag{1}$$

$$\chi^{-1} = N(0) \left[\frac{\delta x}{x_c} + \left(\frac{T}{T_0} \right)^2 + \frac{|\nu|}{T_0} + \left(\frac{v_F q}{T_0} \right)^2 \right], \tag{2}$$

which is valid for T, vq, $|\nu| \ll T_0$. Here T is temperature, $N(0)$ is the density of states at the Fermi surface, ν is Matsubara frequency, q is momentum, v_F is the Fermi velocity, and δx is the deviation of the control parameter x from its critical value x_c, where the transition temperature turns into zero.

We obtained this action by repeating the Abrikosov-Gorkov calculation [5] for an s-wave BCS superconductor doped with magnetic impurities. Apart from the original calculation [5], the frequency and momentum dependence of the disorder-averaged pair propagator was kept and at the end both the pair field and the momentum were rescaled to give (1),(2).

The quantity T_0 is the only characteristic energy scale of the effective action (1),(2). In a BCS superconductor it is of the order of the pair-breaking rate, which at the critical point is of the order of the transition temperature in the clean system. Both quantities are much less than the Fermi energy ϵ_F. However, in a strongly correlated material, T_0 may, in principle, be of the order of ϵ_F.

These two limiting cases correspond to different physics. In the BCS case, the quartic term in (1) can be neglected at all experimentally relevant temperatures T well below T_0. By contrast, if $T_0 \sim \epsilon_F$, the feedback of the quartic term dominates at $T \ll T_0$. In the latter case, one has to regard T_0 as a phenomenological parameter, resulting from a strong coupling, or non-BCS-pairing mechanism.

With (1),(2) at hand, one can calculate various thermodynamic and transport properties of interest. Table I presents the results for the zero-temperature quasiparticle decay rate due to scattering by the Δ field, and for the leading corrections to low-temperature thermodynamics and transport at $T_c = 0$. In the BCS limit, corrections to the specific heat coefficient come from Gaussian fluctuations of the pair field [6]. The correction for a strong coupling case was found in [3] using renormalization group methods. The quasiparticle decay rate due to scattering off pair fluctuations is estimated by the diagram in Fig. 1(a). It is essential for the calculation of conductivity and of the quasiparticle decay rate that the electron vertex corrections are not singular, since pair breaking makes the lifetime of a Cooper pair finite. The leading conductivity correction

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TABLE I. Order of magnitude and leading temperature dependences of the imaginary part of the electron self-energy $\Sigma(\omega; p = p_F; T = 0)$ due to scattering by the fluctuations of Δ, the specific heat coefficient correction $\delta C(T)/T$, and the conductivity correction $\delta\sigma(T)$ in the weak coupling (BCS) limit ($T_0 \ll \epsilon_F$) and in the strong coupling limit ($T_0 \sim \epsilon_F$). The value of σ_0 corresponds to the residual normal state conductivity.

	Electron self-energy $\mathrm{Im}\Sigma(\omega)$	Specific heat coefficient $\delta C(T)/T$	Conductivity correction $\delta\sigma(T)$
$T_0 \ll \epsilon_F$, $d = 3$:	$\frac{(T_0\omega)^{3/2}}{\epsilon_F^2}$	$\frac{T_0^{3/2}T^{1/2}}{\epsilon_F^3}$	$\sigma_0\frac{T_0^{3/2}T^{1/2}}{\epsilon_F^2}$
$T_0 \sim \epsilon_F$, $d = 3$:	$\frac{\omega^{3/2}}{\epsilon_F^{1/2}}$	$\frac{T^{1/2}}{\epsilon_F^{3/2}}$	$\sigma_0\left(\frac{T}{\epsilon_F}\right)^{1/4}$
$T_0 \ll \epsilon_F$, $d = 2$:	$\frac{\omega T_0}{\epsilon_F}$	$\frac{T_0}{\epsilon_F^2}\ln\left(\frac{T_0}{T}\right)$	$\sigma_0\frac{T_0^2}{T\epsilon_F}$
$T_0 \sim \epsilon_F$, $d = 2$:	ω	$\frac{1}{\epsilon_F}\ln\left(\frac{\epsilon_F}{T}\right)$	$\sigma_0\ln\left(\frac{\epsilon_F}{T}\right)$

is given by the Aslamazov-Larkin diagram [7] shown in Fig. 1(b), which can be regarded as conductivity of particles with the inverse propagator given by (2), at $\delta x = 0$.

The calculation was done by renormalization group analysis of the expression for the Aslamazov-Larkin correction in Fig. 1(b). After transforming the Matsubara sum into a contour integral and going to dimensionless variables, it reads [8]

$$\Delta\sigma = \int^1 q^2 d^D q \int^1 \frac{dz}{T}\, \frac{1}{\sinh^2 \frac{z}{2T}}\, [\mathrm{Im}\chi(q, z + i0)]^2, \tag{3}$$

where both the energy and the momentum cutoff have been set to unity. When renormalizing, we will use the following steps [3]: first integrate out a thin outer shell in the momentum space between 1 and $1 - 1/b$. Then rescale the momentum ($q \to qb$) to restore the cutoff, then rescale the energy ($z \to zb^2$), the mass term, and the temperature ($T \to Tb^2$), and, finally, integrate out an energy shell to restore the energy cutoff. At each step the quartic interaction induces corrections to the mass term

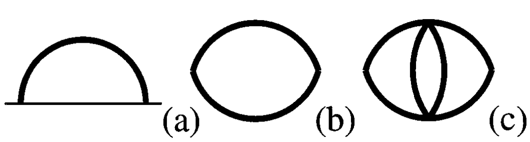

FIG. 1. (a) Graph for the quasiparticle self-energy due to scattering by pair fluctuations. Solid lines denote the pair propagator; thin line denotes the quasiparticle Green function. (b)–(c) Graphs for the conductivity correction: (b) the Aslamazov-Larkin graph; (c) an example of a graph with vertex corrections. Scaling dimension of all such graphs is an integer multiple of $(2 - D)$.

and to itself (see Ref. [3] for details of the renormalization group equations).

As a result, one arrives at the following transformation law for the Aslamazov-Larkin correction:

$$\Delta\sigma[J] = b^{2-D}\Delta\sigma[J'] + \ln b f[J],$$

where J denotes the mass term, the temperature, and the quartic coefficient, J' denotes their renormalized values, and D is the dimensionality of the sample. The precise form of $f[J]$ can be easily obtained using the above renormalization procedure. However, only two features of $f[J]$ are important: (a) as long as the running value of T is smaller than the cutoff, $f[J]$ has rather weak dependence on its arguments; this corresponds to the quantum renormalization region; (b) in the classical renormalization region, when T exceeds the cutoff, $f[J]$ is proportional to T. Thus $\Delta\sigma$ can be written as

$$\Delta\sigma = \int_0^{\ln b^*} dx\, f[J(e^x)]e^{(2-D)x}, \tag{4}$$

where b^* is the value of the rescaling factor at which the mass term reaches the cutoff and the scaling process stops. It is of the order of $T^{-3/4}$ in three dimensions and of the order of $\sqrt{\ln(1/T)/T}$ in two [3]. To evaluate (4), one also needs the value of b_1 such that $T(b_1) \sim 1$, which is $b_1 \sim T^{-1/2}$ independent of the dimensionality. With these prerequisites, the answers in Table I follow as soon as one neglects the dependence of $f[J(e^x)]$ on all the couplings except temperature:

$$f[J(e^x)] = f[Te^{2x}].$$

In three dimensions, the same result can be obtained just by replacing the "bare" T^2 mass term in (2) by its renormalized value $T^{3/2}$, and then calculating the conductivity correction as per (3).

Scaling analysis also allows us to show that the vertex corrections are negligible. Their inclusion reduces to putting into the diagram in Fig. 1(b) extra bubbles such as the one in Fig. 1(c). Each bubble contributes two Green's functions plus an integral over the energy and momentum. After an infinitesimal scaling transformation, this gives an extra factor of b^{2-D}. Thus, in three dimensions, vertex corrections are irrelevant. In two dimensions, they appear marginal yet do not introduce any extra corrections. This can be established by using the Ward identity and writing the renormalization equation directly for the current vertex (see Fig. 2), and then inserting the solution into the corresponding expression for the conductivity.

We discuss theoretical and experimental implications of the results. As envisaged by Hertz [2], under quite general circumstances the superconducting quantum critical point falls into the $z = 2$ universality class. Although in principle different behavior cannot be ruled out, it appears to require additional tuning of parameters, as well as

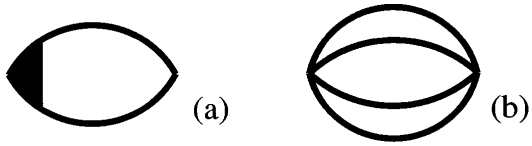

FIG. 2. (a) Graph for the conductivity, including the current vertex, denoted by the dark triangle. (b) Contribution to the free energy in the second order in the quartic term, which gives rise to renormalization of the current vertex.

rather unusual features of the pairing phenomenon itself, such as the gap vanishing at the entire Fermi surface [9].

In a weakly interacting disordered metal, T_0 in (2) is of the order of the impurity scattering rate. In this case anomalous corrections due to quantum criticality in the Landau-Ginzburg regime are of the order of the weak localization corrections [10]. Therefore, in two dimensions our results are fully consistent only as long as all corrections are small and additive. Moreover, in two dimensions, the electron-electron interaction is known to lead to the linear temperature dependence of the quasiparticle decay rate [10,11], regardless of closeness to the quantum critical point. Thus, for a weakly interacting two-dimensional system, the entire normal region corresponds to the two-dimensional disordered metallic regime (see Fig. 3) with the quasiparticle decay rate proportional to the quasiparticle energy.

However, we expect that in a strongly correlated system with suppressed weak localization effects, the superconducting quantum critical point still falls into the $z = 2$ universality class and Table I may describe the actual state of affairs for all dimensionalities. In this case, Fermi liquid turns marginal only at $T_c = 0$ and crosses over to the normal-Fermi-liquid behavior as the system is driven away from the quantum critical point into the metallic phase.

Finally, we comment on the diagnostic opportunities, furnished by measurements at a superconducting quantum critical point. In light of the discussion in the beginning of the paper, one is led to conclude that in a *clean* time reversal invariant system, observation of singular behavior at the superconducting quantum critical point would mark a *very* peculiar phenomenon, as in a clean BCS superconductor suppression of T_c completely eliminates pair fluctuations. Since any sample contains impurities, in reality the above conclusion refers to temperatures above the elastic scattering rate: $T > \tau^{-1}$. The latter can be extracted from the residual resistivity measurements of the sample.

To summarize, we studied the properties of a superconductor near a quantum critical point driven by pair-breaking disorder. Superconducting fluctuations are characterized by a dynamical critical exponent $z = 2$. In a BCS superconductor at experimentally accessible low temperatures, the singular contribution to the conductivity is proportional to $\sqrt{T}$ in three dimensions, but to $1/T$ in two dimensions. In a superconductor with the characteristic energy scale of the order of ϵ_F at the quantum critical point, the contribution to the conductivity varies as $T^{1/4}$ in three dimensions, and as $\ln(1/T)$ in two.

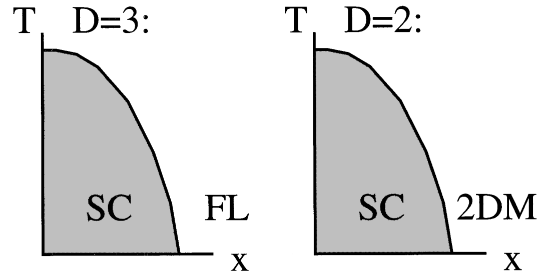

FIG. 3. Schematic phase diagram in three ($D = 3$) and in two ($D = 2$) spatial dimensions. Shaded regions correspond to the superconducting (SC) phase. The normal phase corresponds to the Fermi liquid (FL) in three dimensions, and to a two-dimensional disordered metal (2DM) in two.

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