Fracture of a Brittle Membrane

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The fracture of a brittle membrane by a localized transverse impact is studied using a numerical lattice model. The fracture patterns which are found experimentally [N. Shinkai, in *Fractography of Glass,* edited by R.C. Bradt and R.E. Tressler (Plenum, New York, 1994)] by dropping a heavy ball on a thin glass plate are reproduced. At a very high impact velocity only a small hole is created at the impact area (Hertzian fracture). At lower velocities tangential and radial cracks are formed. In the model calculations, pure Hertzian fracture appears in the limit where inertial forces dominate over the elastic forces, while tangential fracture appears in the opposite limit. Radial cracks are demonstrated to be a consequence of nonlinear deformations or an externally applied in-plane strain. [S0031-9007(97)04464-5]

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Membranes and thin plates have many advantages from a technological point of view and appear in all kinds of man-made structures. In many of the applications, however, membranes are useful only if they are made of a stiff and light material so that they are rigid. Stiff and light materials are easy to find but many of them are also very brittle, and they therefore break easily. Everyday examples of such structures are ceramic and glass plates (like windows). To be able to optimize these structures it is necessary to understand how they fracture. Irrespective of any practical use, it is interesting to study membrane fracture as a physical phenomenon.

In this Letter we report the results of lattice model simulations of a brittle two-dimensional solid that can be deformed in three dimensions. We demonstrate that our model can reproduce the fracture patterns which were found by Shinkai [1] in his experiments on thin glass plates. In these experiments, thin square-shaped glass plates were supported at the edges, and a small and heavy ball was dropped on them from a point above the center of the plate. The fracture patterns that were formed by the impacts typically consisted of three types of cracks. For a high impact velocity, only a small circular hole at the point of impact was formed. This is a so-called Hertzian fracture [1]. At lower velocities, radial and tangential cracks appeared (Fig. 1). The radial cracks were fairly straight and directed outwards from the point of impact, while the tangential cracks formed a more or less circularly symmetric crack with the impact point as the center of the circle. At still lower impact velocities, only radial cracks were formed. Finally, of course, at very low velocities no cracks appeared.

Notice that the chronological order of the appearance of these cracks can be determined from Fig. 1. As all radial cracks originate from the hole at the impact point, the hole must be created before the radial cracks. Similarly, as the tangential cracks have discontinuities at the radial cracks, they must be formed after the radial cracks [1]. The chronological appearance of the cracks in the simulations will be discussed below.

Another, more fundamental, aspect of this work is to develop a computer simulation model of fracture, which can more readily be compared to experimental results than the so far extensively used strictly two-dimensional models. For a review of the two-dimensional models of quasistatic fracture see, e.g., Refs. [2,3], and for results of dynamic fracture of two-dimensional models see, e.g., Refs. [4–9]. The present membrane model also has a two-dimensional geometry, but the difference lies in the possibility to deform in all three space dimensions.

We have chosen to use a beam lattice model for our numerical investigations for two reasons. Beam lattices are efficient from a numerical point of view, and they form a straightforward discretization of a brittle solid obeying “Cosserat elasticity” [10,11]. That is, large scale rotations are possible. The particular lattice we use here is a triangular lattice with beams as the lattice bonds.

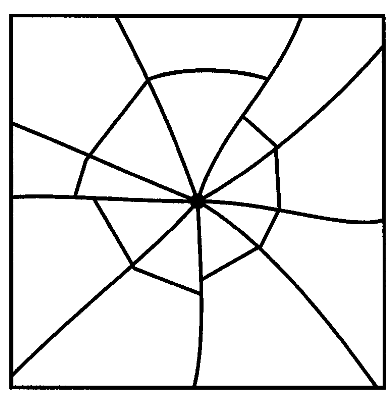

FIG. 1. Schematic picture of the fracture pattern on a thin glass plate. Hertzian type fracture at the center of the plate, radial cracks directed outwards from the center, and a roughly circular tangential crack around the center. Sketched after Ref. [1].

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The beams, which are assumed to have no mass, connect lattice sites at which masses are located. The beams are assumed to have a square cross section, and we use the stiffness matrix of a slender beam (i.e., bending dominates over shear deformations), which can be derived from linear theory of elasticity [12]. Notice that we only use linear elasticity, and therefore neglect higher order terms in the displacements of the sites which arise from deformations of the membrane. In other words, our model is correct, in a strict sense, only for infinitesimal deformations. The length, the cross-section area, and Young's modulus of the beams are all, for the sake of simplicity, set to unity. The Poisson ratio is assumed to be zero. These are not actually "slender" beams, but we are only interested in the qualitative behavior of the model, and therefore we choose the simplest possible set of parameter values. With these parameter values, the forces needed to elongate or shear a beam by a unit distance are both unity. The angular momentum needed to create a unit torsional rotation and a unit angular rotation of one of the ends of a beam are $1/12$ and $1/3$, respectively. The masses and the moments of inertia of the sites are also both set to unity. In the simulations we use lattices of 50×70 sites located in the xy plane, while the impact is in the z direction. To study the effect of the thickness of the membrane, we also simulate a two-dimensional lattice in the xz plane. Notice that a two-dimensional lattice is sufficient in this case if we assume circular symmetry, i.e., we simulate a circular instead of a square plate. Then all deformations of the lattice will only depend on the radial distance from the center of the impact point, and it suffices to study a two-dimensional cross section of the lattice. In this case we use lattices of 150×10 sites. The dynamics of the lattices are calculated using a discrete form of Newton's equations of motion including a small linear viscous dissipation term,

$$\left[\frac{M}{\Delta t^2} + \frac{C}{2\Delta t}\right] U(t + \Delta t) = \left[\frac{2M}{\Delta t^2} - K\right] U(t) - \left[\frac{M}{\Delta t^2} - \frac{C}{2\Delta t}\right] \times U(t - \Delta t), \tag{1}$$

where M is a diagonal mass matrix, K the stiffness matrix, C a diagonal damping matrix, U a vector containing the displacements of the sites from their equilibrium positions, Δt the length of the discrete time step, and t the time. In the simulations, both M and C are set proportional to the unity matrix. The boundary conditions imposed on the lattice in the xy plane are such that the sites at the boundaries of the lattice are constrained to remain at their original positions, while the sites in a circular area in the middle of the lattice are forced to move a distance $-vt$ in the z direction. The lattice is set to be at static equilibrium at the time $t = 0$, and the locations of all sites at any time can be calculated iteratively using Eq. (1). For the lattice in the xz plane only the sites at the left and right edges of the lattice are constrained to remain at their original positions, while a number of sites in the middle of the upper boundary move in the negative z direction. To include fracture, we set a threshold value for the elongation of a beam at which failure is irreversibly initiated. Fracture is, however, not instantaneous. At the threshold we assume that the Young's modulus of a beam begins to decrease linearly in time until it reaches zero. The rate at which the modulus decreases is a parameter (α) for which we have tested different values. For a material like glass, for example, bending would be a more physical breaking criterion than elongation. We have also tested a combined breaking criterion of elongation and out-of-plane bending of the membrane. The crack patterns found were qualitatively similar to those found using only the elongation criterion. Thus, for simplicity, we use only the elongation criterion. At this point it should be mentioned that our lattice model mimics the deformations of a solid correctly on a scale larger than a lattice bond. Fracture, however, is a local process and must therefore be defined on the smallest possible scale on the lattice (i.e., on the scale of a single bond). On this scale the geometry of the lattice will affect the direction of a crack. This has no physical counterpart in real world, at least not for an isotropic material. The effect of the lattice geometry will be discussed more below.

In Fig. 2 we show a sequence of snapshots of a simulation of an impact. The radius of the impact is 5, $\Delta t = 0.05$, $\alpha = 0.05/\Delta t$, and $v = 1/600\Delta t$. The snapshots are taken after 200, 400, and 600 time steps, respectively, and the lattice is shown tilted at angles $\pi/4$ and $\pi/2$. The figure clearly shows the formation of a circular well. Fracture is initiated at the points where the walls of the well are the steepest, and eventually a circular crack is formed around the center of the impact point, that is, a tangential crack. This figure also displays the effect of lattice geometry. In the snapshot taken at 400 time steps, it can be seen that the tangential crack is first formed at the location where a lattice bond is in the direction of the gradient of the slope of the well. From this figure it is also evident that the lattice is not at equilibrium during the impact, and that dynamic effects are important. With these parameter values, the radius of the tangential crack is rather small. If the velocity of the impact is smaller, the deformation of the lattice reminds us more of a lattice at equilibrium. This is demonstrated in Fig. 3 where we have used the same lattice as in Fig. 2, but with $v = 1/8000\Delta t$. In Fig. 3 the deformations are so close to those at equilibrium that dynamical forces are negligible as compared to elastic forces, and the tangential crack has a bigger radius. The same phenomenon can be seen in simulations of a lattice in the xz plane. This is demonstrated in Fig. 4. In Fig. 4(a), the velocity of the impact is $v = 1/200\Delta t$, while it is $v = 1/8000\Delta t$ in Fig. 4(b). In Fig. 4(a) only a small hole is formed at the

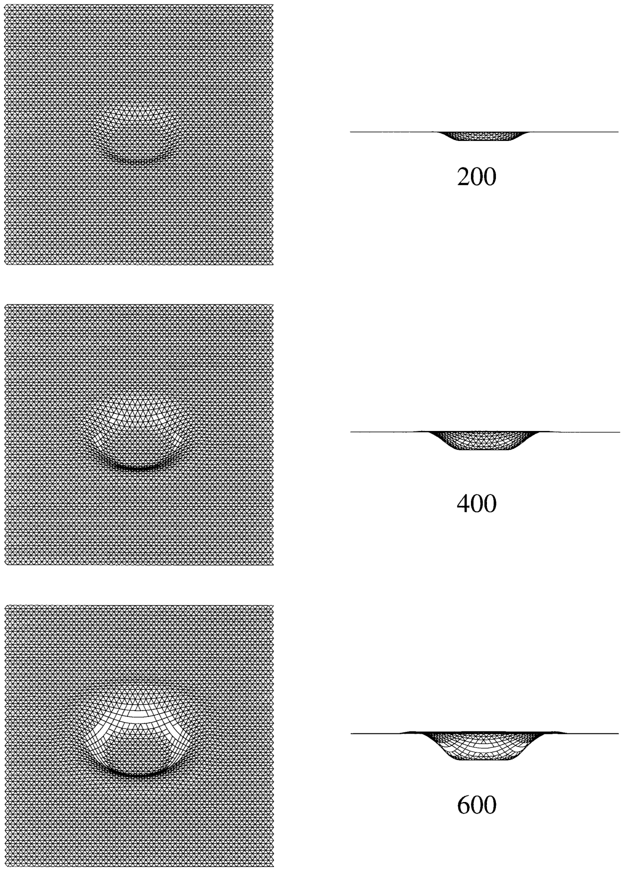

FIG. 2. A sequence of snapshots of a simulation of an impact. The radius of the impact is 5 bond lengths. The impact object is not shown in the figure. The snapshots are taken after 200, 400, and 600 time steps, respectively, and the lattice is shown tilted at angles $\pi/4$ and $\pi/2$. Broken bonds are removed.

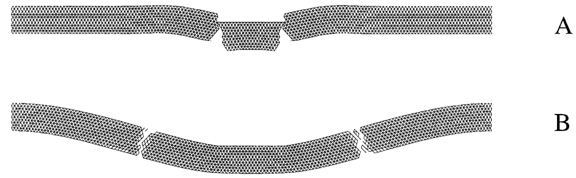

FIG. 4. Deformations of a cross section of a membrane with a finite thickness. (a) Hertzian fracture at a high impact velocity ($v = 1/200\Delta t$), and (b) tangential cracks at a low impact velocity ($v = 1/8000\Delta t$).

location of the impact, i.e., a Hertzian fracture, while a tangential crack is formed in Fig. 4(b).

Based on the results shown in Figs. 2–4, it is clear that our model reproduces either a Hertzian fracture or tangential cracks, depending on the velocity of the impact, in agreement with the experiments [1]. So far, however, our model has not reproduced the radial cracks. Looking at Fig. 1, one might intuitively expect that the radial cracks appear as small cracks initiated at impurities close to the point of impact. Because of the in-plane strain resulting from bending of the membrane, such cracks would propagate as a consequence of stress enhancement at their sharp tips. If the slope of the deformation well (cf. Fig. 2) is dU/L (i.e., dU is the displacement difference in the z direction of the ends of a bond, and L is the bond length), then the *in-plane* strain σ_{xy} at that point will simply be

$$\sigma_{xy} = 1/2(dU/L)^2 + O((dU/L)^4), \qquad (2)$$

which obviously cannot be accounted for by our linear model. To test our assumption that the radial cracks appear as a consequence of in-plane strain, we therefore apply an extra, constant, in-plane strain at $t = 0$, and keep the boundaries of the lattice at the new locations during the impact simulation. This means that at $t = 0$ we adjust the displacements of all sites so that the entire lattice is at equilibrium with the new, nonzero strain, boundary conditions. The result is demonstrated in Fig. 5, which obviously displays radial cracks originating from the Hertzian type crack formed at the impact. In Fig. 5,

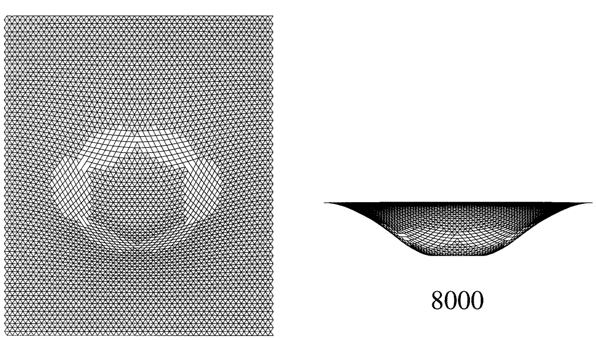

FIG. 3. Deformation of the lattice of Fig. 2 after 8000 time steps with an almost 15 times lower impact velocity. The tilt angle of the lattice is $0.75\pi/2$.

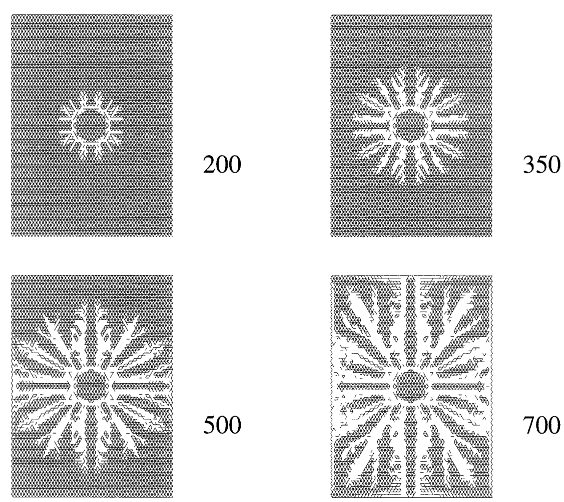

FIG. 5. Radial cracks in a lattice with an externally applied in-plane strain: $\alpha = 0.1/\Delta t$, $v = 1/600\Delta t$, and the in-plane strain is $\sigma_{xy} = 0.167$. Snapshots at 200, 350, 500, and 700 time steps are shown. The tilt angle of the lattice is zero.

$\alpha = 0.1/\Delta t$, $\upsilon = 1/600\Delta t$, and the in-plane strain is $\sigma_{xy} = 0.167$.

To study the chronological appearance of the tangential and radial cracks, model parameters would have to be found for which both these types of cracks appear. Such parameters proved to be difficult to find. Much larger systems and longer computing times than available at the moment would probably be needed to achieve this.

Finally, one could try to find parameter values which match a specific material and compare the simulation results with experiment. This is, however, not very easy as the correct postfracture elastic behavior (i.e., the parameter α in our model) is difficult to simulate correctly. This is an important issue as it seems to have quite a large influence on the final crack pattern [a large α (i.e., $\alpha \approx 1/\Delta t$) leads to few and narrow cracks, while a very small α leads to shattering of the entire membrane]. In experiments, a glass plate can remain as one piece even if it contains several cracks because the crack surfaces are rough, and therefore interlocking of the fragments will hinder the plate from falling apart. Thus, the plate will behave almost as if it were undamaged, at least with respect to compressive deformations, although it is fractured. In the present lattice model, we have only used the simple linear decay of the Young's modulus as the postfracture behavior of bonds. This feature should be developed further before any quantitative comparison between simulations and experiment will sensibly be done. Alternatively, the membrane could be simulated as a full three-dimensional model, which allows for interlocking of fragments, and which accounts for the delay in crack formation as cracks have to penetrate through the membrane in the z direction. Another aspect, which should be considered before any detailed comparisons of simulations and experiments, is the appearance of impurities in all real materials. Based on earlier studies [2,3] such impurities are important in fracture. To carry out these simulations would, however, require very large lattices and long computing times, and would be outside the scope of this analysis.

In summary, we have demonstrated that a numerical lattice model of a brittle membrane can reproduce all three types of cracks found in experiments on glass plates, i.e., Hertzian type of fracture, radial cracks, and tangential cracks. At high impact velocities only the Hertzian fracture appears, while at lower velocities tangential cracks appear. This is consistent with experiments. The radial cracks only appear in a nonlinear model or if an extra in-plane strain is applied. Even though the present lattice model excellently mimics membrane fracture in a qualitative sense, further refinements have to be done to the model before quantitative comparisons with experiment are possible.

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