Normal Modes of the $B = 4$ Skyrme Soliton

Chris Barnes,[1,*] Kim Baskerville,[2,†] and Neil Turok[3,‡]
[1] Joseph Henry Laboratory, Princeton University, Princeton, New Jersey 08540
[2] Physics Department, University of Wales Swansea, Singleton Park, Swansea SA2 8PP, United Kingdom
[3] DAMTP, Silver Street, Cambridge CB3 9EW, United Kingdom
(Received 1 April 1997)

The Skyrme model of nuclear physics requires quantization if it is to match observed nuclear properties. A simple technique is used to find the normal mode spectrum of the baryon number $B = 4$ Skyrme soliton. We find 16 vibrational modes and classify them under the cubic symmetry group O_h of the static solution. The spectrum possesses a remarkable structure, with the lowest energy modes lying in those representations expected from an approximate correspondence between Skyrmions and Bogomolny-Prasad-Sommerfeld monopoles. The next mode up is the "breather", and above that are higher multipole breathing modes. [S0031-9007(97)03613-2]

PACS numbers: 24.85.+p, 11.10.Lm, 12.39.Dc, 27.10.+h

In the Skyrme model of nuclear physics, both pions and nucleons are represented by a single scalar SU(2) group valued field, $U(x)$. Pions occur as field quanta, while baryons are instead represented as topological solitons. The classical Skyrme theory, with a simple quantization of the spin and isospin collective modes, provides a description of nucleons and the Δ resonance in modest agreement with experiment [1,2].

Applying the Skyrme model to larger nuclei and to nuclear matter is an even more interesting proposition, since with no additional free parameters one could compare the theory with the binding energies and gamma ray spectra of all nuclei. There has been progress in understanding the structure of Skyrme multisolitons [3,4], but it is clear that unless quantum fluctuations about the static solutions are included, there is little chance of success. For real nuclei the relative kinetic energies of the nucleons in the ground state are large, so a quantization at least of a number of degrees of freedom equal to the number of nucleon coordinates is essential. Note that the simple collective coordinate quantizations of spin and isospin [1,2] include effects of order $\hbar^2$, while ignoring effects of order $\hbar$.

Recently there has been some progress in quantization, based upon Manton's notion of representing low energy solitonic excitations as motion on a finite dimensional space of moduli. A study of the deuteron by Leese *et al.* [5] using an instanton approximation for the field configurations, gave encouraging agreement with experimental properties, but only included two of the four expected vibrational modes of the deuteron. Walet [6] extended this treatment to estimate all vibrational frequencies for the deuteron and triton, again using the instanton approximation.

In this Letter we follow a different track. We directly compute the low energy normal modes, finding their frequencies and the representations they lie in under the static soliton's symmetry group. The frequencies and representations provide a coordinate-independent description of the configuration space around the static solution; in the harmonic approximation they determine the quantum vibrational energy levels. Our results provide new insight to the moduli space approach, since the representations for the lowest frequency modes turn out to be just those expected from a recently understood approximate correspondence between Skyrmions and Bogomolny-Prasad-Sommerfeld (BPS) monopoles [7]. Furthermore, the success of these calculations allows one to contemplate going beyond the moduli space approximation and performing a full semiclassical quantization of the field theory. This is an attractive goal, since the Skyrme theory could then be incorporated in the framework of chiral effective Lagrangians [8], allowing a unified treatment of mesons, baryons, and higher nuclei.

I Method.—Since the SU(2) manifold is a 3-sphere, we represent it in terms of a scalar field $\phi \in \mathbf{R}^4$, with $\phi^a\phi^a = 1$. In terms of this field, the Skyrme Lagrangian density is

$$\begin{aligned}\mathcal{L} = \tfrac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi + \omega_\pi^2\phi^1 + \lambda(\phi\phi - 1) \\ + \tfrac{1}{4}\{(\partial_\mu\phi\,\partial_\nu\phi)(\partial^\mu\phi\,\partial^\nu\phi) \\ - (\partial_\mu\phi\,\partial^\mu\phi)(\partial_\nu\phi\,\partial^\nu\phi)\},\end{aligned} \tag{1}$$

with λ a Lagrange multiplier field. Here, length and time are in units of $\frac{2}{eF_\pi}$, energy in units of $\frac{F_\pi}{2e}$. In these rescaled units, the only remaining parameter is $\omega_\pi = \frac{2m_\pi}{eF_\pi}$, the oscillation frequency for the homogenous pion field. For the most part we have adopted the "standard" value [2] of $\omega_\pi = 0.526$, although we have also performed calculations at twice this value.

The mixed space and time derivative terms in the Skyrme Lagrangian make numerical solution in general difficult. Isolating the time derivative terms one has

$$\mathcal{L} = \frac{1}{2}\,\dot\phi^a K^{ab}(\partial_i\phi)\dot\phi^b - V(\phi, \partial_i\phi) + \lambda(\phi\phi - 1), \tag{2}$$

where $K^{ab} = \delta^{ab}[1 + (\partial_i\phi)^2] - \partial_i\phi^a\partial_i\phi^b$ is a local inertia matrix, and $V(\phi, \partial_i\phi)$ is the potential. In general,

0031-9007/97/79(3)/367(4)$10.00

K^{ab} is time dependent, but for small perturbations around a static solution $\phi_{\rm st}(\boldsymbol{x})$ we write $\phi(\boldsymbol{x},t) = \phi_{\rm st}(\boldsymbol{x}) + \epsilon(\boldsymbol{x},t)$, $\epsilon \ll 1$ and to second order in ϵ the Lagrangian is

$$\mathcal{L} = \frac{1}{2}\dot{\epsilon}^a K^{ab}(\partial\phi_{\rm st})\dot{\epsilon}^b - V(\phi, \partial_i\phi) + \lambda(\phi\phi - 1). \tag{3}$$

This Lagrangian leads to classical field equations:

$$K^{ab}(\partial\phi_{\rm st})\ddot{\phi}^a = \partial_i\left(\frac{\partial V}{\partial\phi^b{}_{,i}}\right) - \frac{\partial V}{\partial\phi^b} + \lambda\phi^b, \tag{4}$$

where the matrix $K^{ab}(\boldsymbol{x})$ is taken to be its value at the static classical solution. Equation (4) closely approximates the Skyrme equations for fields near a static classical solution, precisely the desired regime for studying soliton normal modes.

In order to numerically solve the field equations, we discretize the action, (3), using a diagonal differencing scheme for the four spatial derivative terms, achieving a high degree of locality and second order accuracy in both spatial and time steps. The numerical code conserves energy and baryon number to within 1 part in 10^5 over the course of extremely long (50k time step) runs. Periodic boundary conditions are used. We first create the appropriate minimal energy static solution by straightforward time evolution from four-Skyrmion initial conditions. A simple relaxation procedure sets the field momenta zero each time the kinetic energy reaches a maximum. We find the fields rapidly converge on the minimum energy configuration. K^{ab} is set equal to δ^{ab} in this part of the calculation, since this does not affect the final static solution.

Next we slightly perturb the fields and evolve them forward again, but now using the full inertia matrix $K^{ab}(\partial\phi_{\rm st})$. The evolving field is

$$\phi(\boldsymbol{x},t) = \phi_{\rm st}(\boldsymbol{x}) + \sum_{\rm modes}\epsilon_n\delta_n(\boldsymbol{x})\cos(\omega_n t) + \boldsymbol{O}(\epsilon^2), \tag{5}$$

where the functions $\delta_n(\boldsymbol{x}) \in \mathbf{R}^4$, obeying $\delta_n(\boldsymbol{x})\phi_{\rm st}(\boldsymbol{x}) = 0$, are the normal modes, each excited with amplitude ϵ_n. The normal mode frequencies ω_n are found by Fourier transforming $\phi(\boldsymbol{x},t)$ with respect to time at any point in the box, and plotting the resulting power spectrum.

The space of perturbations has a useful inner product

$$\langle\delta_1|\delta_2\rangle = \int_{\rm box}\delta_1^a(\boldsymbol{x})K^{ab}(\boldsymbol{x})\delta_2^b(\boldsymbol{x})\,d^3\boldsymbol{x}, \tag{6}$$

which is zero for normal modes δ_1 and δ_2 if $\omega_1 \neq \omega_2$. The inner product allows one to determine the degeneracies of the normal mode frequencies and the representations in which the modes transform under the soliton's symmetry group.

II Results for the $B = 4$ soliton.—We have applied this technique to the case of the Skyrme soliton with baryon number four. The $B = 4$ configuration provides an especially simple case for quantization, because the ground state possesses zero angular momentum and isospin. The static soliton has cubic symmetry; its energy and baryon number density concentrate along the edges of a cube [4]. The full 48 dimensional cubic group of symmetries O_h (for notation see [9]) is generated by 90° and 120° rotations, and parity I. After an appropriate global isospin rotation, the action of these group elements on spatial coordinates and pion fields $[\phi^a = (\sigma, \vec{\pi})]$ is as follows [10]:

$$C_4:\quad (\pi^1,\pi^2,\pi^3)(x,y,z) \to (-\pi^2,-\pi^1,-\pi^3)(-y,x,z),$$
$$C_3:\quad (\pi^1,\pi^2,\pi^3)(x,y,z) \to (\pi^2,\pi^3,\pi^1)(y,z,x),$$
$$I:\quad (\pi^1,\pi^2,\pi^3)(x,y,z) \to (\tilde{\pi}^1,\tilde{\pi}^2,\tilde{\pi}^3)(-x,-y,-z),$$

with $\tilde{\pi}^1 = \frac{1}{3}(\pi^1 - 2\pi^2 - 2\pi^3)$, etc. From this it is straightforward to check that a homogeneous pion field falls into the two dimensional representation E^+ and the one dimensional representation A_2^-, where superscripts indicate parity.

A typical power spectrum of the perturbations is shown in Fig. 1. Spectra at different sites and for different field components show the same peaks, but with differing heights. Once the normal mode frequencies ω_n are identified, maps of the normal modes $\delta_n(\boldsymbol{x})$ may be constructed by performing discrete Fourier sums on each component of the field as it evolves. For degenerate modes, each set of perturbed initial conditions gives a different linear combination $\sum_i \epsilon_i\delta_i(\boldsymbol{x})$ of modes with the same frequency. Other linear combinations may also be produced by applying the symmetries of the static soliton $S_1, S_2, \ldots$ to a given mode δ. The degeneracy of a given frequency is found by computing the rank of the matrix of inner products between the different linear combinations of degenerate modes produced in these ways. Once the degeneracy is determined, a complete orthonormal basis of modes with this frequency can be constructed. The character of any O_h group element can then be computed

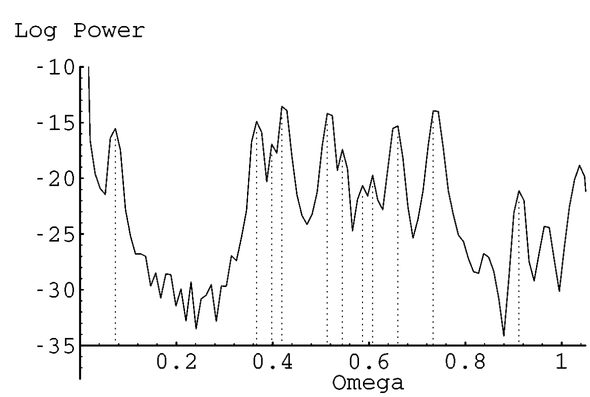

FIG. 1. Fourier power spectrum for perturbations around a $B = 4$ soliton. Frequency is in Skyrme units, power scale is arbitrary. Note that the frequency of homogeneous pion oscillations is $\omega_\pi = 0.526$ in these units.

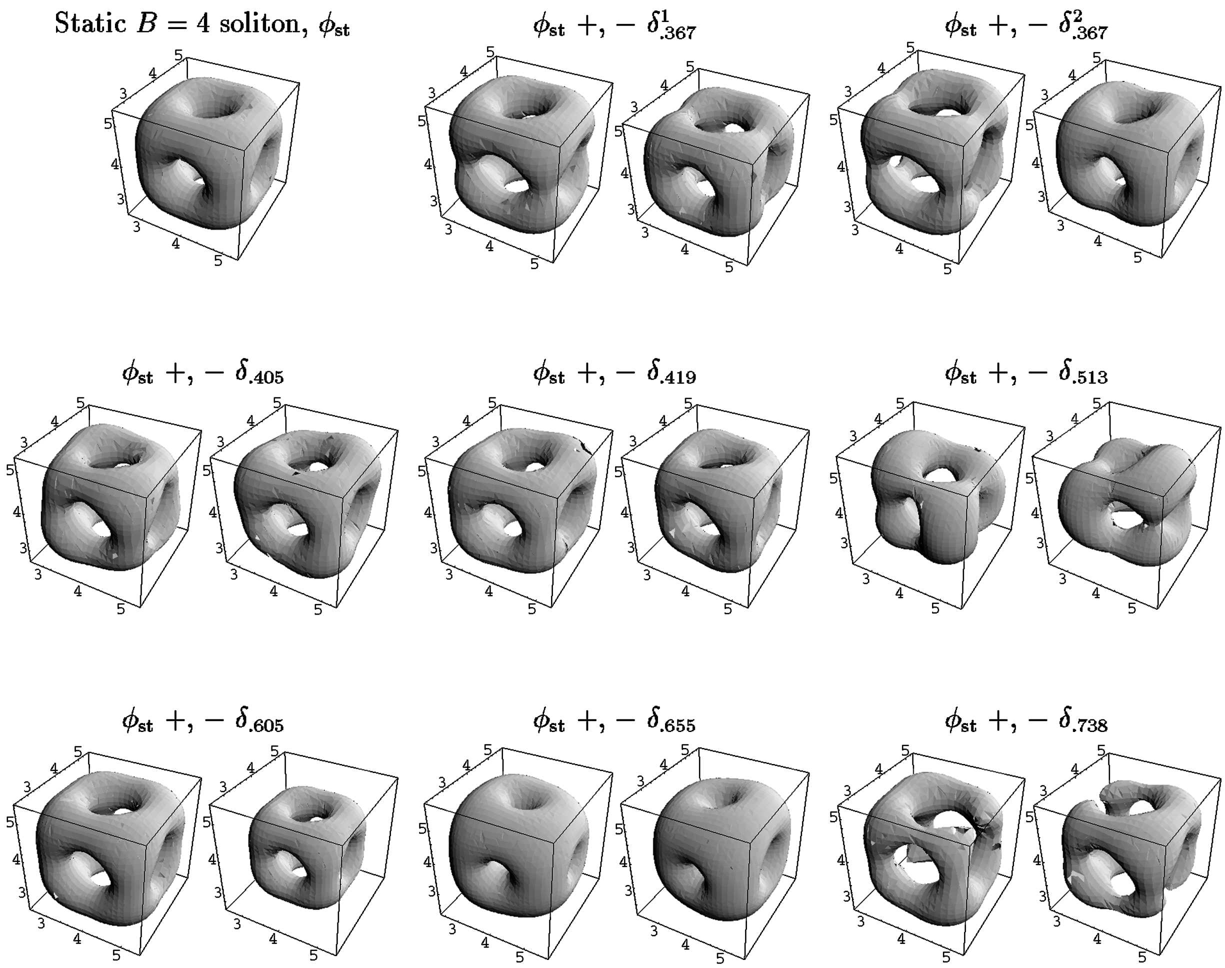

FIG. 2. Contours of constant baryon density for the $B = 4$ soliton, combined with its normal modes, as indexed by their frequencies in Fig. 1. These modes were studied in a box of size $8 \times 8 \times 8$, with a grid spacing $\Delta_x = \frac{1}{8}$ in Skyrme units. For comparison, the soliton itself is a cube roughly $2 \times 2 \times 2$ in these units. For the case $\omega = 0.367$ two orthogonal modes of the degenerate multiplet are shown, in order cases a single mode only is shown.

as a trace. These characters were within ± 0.001 of an integer value, and interpretation was unambiguous.

Each peak in Fig. 1 marks the frequency of a normal mode. The lowest peak (at $\omega = 0.07$) is the rotational zero mode, shifted to nonzero frequency by finite size effects, effectively through interactions with image solitons one box length away. There are also several peaks corresponding to relatively delocalized modes which we interpret as pion radiation. Two (at $\omega = 0.545, 0.587$) correspond to the lowest radiation modes, homogeneous away from the soliton, whose threefold degeneracy is split into $E^+ + A_2^-$ by the presence of the soliton. The first radiation mode at nonzero wave number is at $\omega = 0.908$. The remaining modes are the true vibrational excitations of the α particle. Somewhat fortuitously, the box size was small enough that the lowest inhomogeneous radiation mode has a frequency above the highest vibrational mode.

Four widely separated $B = 1$ Skyrmions have 24 zero modes, corresponding to three translations and three isorotations each. As they combine to form the $B = 4$ soliton, nine of them (three translations, three rotations, and three isorotations) remain as zero modes of the new system. One might expect the remaining 15 modes would survive as low energy vibrational modes. This in one fewer than what we find: we have an additional breather mode.

The vibrational modes distort the $B = 4$ soliton as illustrated in Fig. 2 and explained in the Table I. The modes naturally divide into two sets. The lower nine vibrational modes consist of deformations which, roughly, involve incompressible flow of the baryon charge. In contrast, the higher seven vibrational modes all have a "breathing" character, in which local baryon charge expands or contracts to occupy a greater or lesser volume. The breather itself is simply a rescaling of the size of the soliton, with consequent change in density. The next mode up involves breathing motion of a dipole character, and the one above that of a quadrupole nature.

Remarkably, the vibrational modes below the breather fall into representations corresponding to those for small zero-mode deformations of the BPS four-monopole

TABLE I. Description of the Skyrme field $B = 4$ normal modes marked in Fig. 1. The notation of Hamermesh [9] is used for the representations of the cubic group O_h; superscripts denote parity.

Frequency	Degeneracy	Symmetry	Description
0.07	3	F_1^+	Rotations of the soliton. This is a zero mode broken by the finite box size.
0.367	2	E^+	Lowest vibrational modes. One mode, δ^1, alternately pulls the $B = 4$ cube into two $B = 2$ donuts in two perpendicular directions. The orthorgonal mode, δ^2, pulls it into four $B = 1$ edges one way and two $B = 2$ donuts the other.
0.405	1	A_2^-	The corners of the cube make two interlacing tetrahedra. This mode pulls one tetrahedron out into four $B = 1$ corners, pushing the other one in.
0.419	3	F_2^+	Deform two opposite faces of the cube into rhombuses.
0.513	3	F_2^-	Deform the cube by pulling two opposite edges one one face, and the two perpendicular edges on the opposite face. This takes the cube to four $B = 1$ edges.
0.545	2	E^+	Two of the pion $k = 0$ modes.
0.587	1	A_2^-	The remaining pion $k = 0$ mode, with tetrahedral symmetry.
0.605	1	A_1^+	The breathing mode, with the full cubic symmetry of the soliton.
0.655	3	F_1^-	One face of the cube inflates, while the opposite face deflates.
0.738	3	F_2^+	One pair of diagonally opposite edges inflates; the parallel pair deflates.
0.908	3		The lowest nonzero $(k = 1, 0, 0)$ pion radiative mode.

solution [7]. The same phenomenon occurs in the deuteron [11], and it will be straightforward to check the $B = 3, 5, 6, 7$ solitons using the methods described here. Qualitative similarities between Skyrme multisolitons and BPS multimonopoles and their scattering dynamics have been noted before [3,12]. Our findings suggests a connection between the lowest energy Skyrmion vibrational modes and the multimonopole moduli spaces. If it holds for higher nuclei, there will be $4B - 7$ such modes. It would be very interesting to interpret this number in terms of individual nucleon degrees of freedom (presumably translations and spin and/or isospin).

Finally, we have investigated the effect of doubling the parameter ω_π on the spectrum of modes. All modes move up in frequency. The nine lowest modes move up by (15–25)%, the breathing modes by (30–45)%, and the homogeneous pion modes roughly double in frequency. So as the pion mass in increased from zero, the homogeneous pion mode frequency moves up through the vibrational mode spectrum.

The work described here is a small step towards a full semiclassical quantization of the $B = 4$ Skyrme soliton. There are three directions for future work. First, the normal modes we have found can be used as initial data to search for nonlinear periodic solutions. Second, our techniques in principle allow a computation of the full perturbation spectrum, and thus the soliton's Casimir energy. Finally, we note that the second four-derivative term in the SU(2) Skyrme model, indicated to be present in chiral perturbation theory fits [8], can be straightforwardly included (as can terms involving vector mesons explicitly). Work in each of these directions will likely be needed before a realistic attempt can be made to describe the α particle and its excited states.

We thank Richard Battye, Guy Moore, Conor Houghton, Paul Sutcliffe, and especially Nick Manton for helpful discussions. We also acknowledge the Pittsburgh Supercomputing Center Grant No. AST9G3P.

*Electronic address: barnes@puhep1.princeton.edu
†Electronic address: phkim@python.swan.ac.uk
‡Electronic address: N.G.Turok@amtp.cam.ac.uk

[1] G. Adkins, C. R. Nappi, and E. Witten, Nucl. Phys. **B228**, 552 (1983).
[2] G. Adkins and C. R. Nappi, Nucl. Phys. **B233**, 109 (1984).
[3] R. Battye and P. Sutcliffe, Imperial Report No. 96-97/18, hep-th/9702089 (to be published).
[4] E. Braaten, S. Townsend, and L. Carson, Phys. Lett. B **235**, 147 (1990).
[5] R. A. Leese, N. S. Manton, and B. J. Schroers, Nucl. Phys. **B442**, 228 (1995).
[6] N. Walet, Nucl. Phys. **A606**, 429 (1996).
[7] N. S. Manton, (private communication) noticed the correspondence between the representations of the proper cubic group we found with those for small deformations of the cubic four-monopole BPS solution. One of us (N. T.) independently confirmed that the correspondence also holds for the parity assigments. See also C. J. Houghton, N. S. Manton and P. M. Sutcliffe (to be published).
[8] For a review, see *Dynamics of the Standard Model*, edited by J. F. Donoghue, E. Golowich, and B. R. Holstein (Cambridge University Press, Cambridge, England, 1992).
[9] M. Hamermesh, *Group Theory and its Application to Physical Problems* (Dover, New York, 1962).
[10] R. A. Leese and N. S. Manton, Nucl. Phys. **A572**, 575 (1994).
[11] C. Barnes, K. Baskerville, and N. Turok, e-print hep-th/9704028 (to be published).
[12] R. Battye and P. Sutcliffe, Phys. Lett. B **391**, 150 (1997).