Symmetric Skyrmions

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We present candidates for the global minimum energy solitons of charge 5 to 9 in the Skyrme model generated using sophisticated numerical algorithms. The solitons found are particularly symmetric; for example, the charge seven skyrmion has icosahedral symmetry, and the shapes are shown to fit a remarkable sequence defined by a geometric energy minimization (GEM) rule. [S0031-9007(97)03638-7]

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Skyrmions are topological soliton solutions of a classical nonlinear field theory known as the Skyrme model [1]. As examples of three-dimensional solitonic structures, they are of inherent interest, but they may also have applications in nuclear physics. This arises since the Skyrme model can be obtained as a low energy limit of QCD in the large N_c limit [2], with baryons being identified as the quantized states of the classical soliton solutions [3]. Although this application, which we comment on briefly in the final paragraph, is a motivation for our work, we believe that our findings on the structure of multisolitons is an important step in understanding three-dimensional soliton phenomenology which may be of relevance in a number of branches of theoretical physics.

In a recent Letter [4], we presented the first results from a code which evolves the dynamical equations of motion, exhibiting the dynamics of symmetric configurations in the attractive channel and also the minima for charges 1 to 4. These minima were already known from numerical relaxation calculations [5] and also from instanton calculations [6]. In this Letter, we present what we describe as candidate minima for charges 5 to 9, which we firmly believe are, in fact, the global minima. These minima can be classified in terms of the isosurfaces of baryon density (or energy density), which are seen to fit a remarkable pattern, obeying what we shall call a geometric energy minimization (GEM) rule. We also calculate accurately the energy and average size of each soliton. As in Ref. [4], we shall assume the pion mass is zero, with results for a physical value of the pion mass presented in future work. We do not expect that inclusion of a finite pion mass will effect the overall shape and symmetry of the soliton; rather it will just modify its energy and size.

The full details of the numerical methods are presented in Ref. [7]. The basic procedure is to set up discretized initial conditions for static low energy configurations using either a simple ansatz for a single Skyrmion and the product ansatz, or by calculating the holonomy of instantons [8], or a combination of both, as is the case for the higher charge configurations. The critical feature of the specific configurations chosen is that the Skyrmions are in an attractive or nearly attractive channel. These configurations are then evolved using the discretized equations of motion. As the dynamics proceeds, the system oscillates between maxima and minima of the potential energy. If one stops the dynamics at a minimum of the potential energy and then removes all the kinetic energy from the system, the solution will gradually move towards a local minimum of the system. Once the system is close to the minimum one can also incorporate dissipation which will speed up the process of relaxation. Of course, as with any numerical minimization procedure, one cannot be sure that one has found the global minimum. However, given sufficiently asymmetric, but attractive, initial conditions, as created using the product ansatz, it is likely that one will locate the global minimum.

For charges 1 to 4 it was possible to construct configurations using just the product ansatz, in which all the Skyrmions are in a mutually attractive channel [4,7]. It is not possible to use this naive approach to construct a maximally attractive channel of charge five or higher, and therefore one must consider other approaches. In order for our algorithm to relax quickly to the minimum, we must construct a configuration with low energy, in which most of the Skyrmions are attractive. It need not be the maximally attractive channel, but the algorithm will relax to the minimum quicker if it is close to the most attractive. Some of the configurations chosen relax quicker than others, suggesting that in some cases we have selected the correct configuration and in others we have found one which works, but is perhaps not the best.

We find that it is best to mix the instanton approach for a known symmetric configuration, with a small number of single Skyrmions added using the product ansatz to break the exact symmetries. For example, in the case of $B = 5$, it is possible to construct a highly attractive configuration by adding in two single Skyrmions either side of a $B = 3$ tetrahedron. Since the pion fields of the tetrahedron are similar to those of an anti-Skyrmion at large distances, the configuration is of low energy and relaxes extremely quickly ($\sim$1000 time steps) to the minimum. Using the analogy, to $B = 2$ to $B = 4$ scattering, we tried constructing $B = 6$ and $B = 7$ configurations by surrounding a $B = 3$ tetrahedron with Skyrmions in cyclic and tetrahedral configurations, respectively, using the product ansatz. These configurations relaxed toward the minimum in over 5000 time steps, suggesting that, in fact, these are not the maximally attractive channel. Nonetheless, they are of reasonably low energy and work eventually. For $B = 6$, one can relax to the minimum much more quickly by colliding two tetrahedra. Reassuringly, it is the same minima as calculated before, providing a useful consistency check. In order to calculate configurations of higher charge ($B > 7$), one can just add in a single Skyrmion to the known minima of charge 1 less.

A useful way to represent a Skyrmion is by displaying a surface of constant baryon density. In Fig. 1 we display isosurfaces for the Skyrmions of charges 5 to 9 (1 to 4 are presented in Ref. [4]), using the same constant value for the baryon density in each case, to ensure that the relative sizes of the Skyrmions are accurately represented. The baryon density has maxima at several points in space, which we can think of as vertices of a solid, and these are connected by links of slightly lower baryon density, which can be regarded as the edges of the solid. Clearly in this way we can assign a solid to each Skyrmion which accurately reflects the shape and symmetry of the Skyrmion. In fact, for all the Skyrmions we consider, the associated solid is remarkably close to being composed of regular or nearly regular polygons with a fixed edge length. Given this fascinating result, we shall describe each solid in terms of its construction from polygons. Included in Fig. 1 alongside each baryon density plot is a photograph of the associated solid, constructed using a molecular model building kit. In Table I we list the number of faces of each associated solid, together with the number of each type of polygon from which it is constructed.

It is perhaps useful to give a brief description of each of the solids, which, in conjunction with Fig. 1, should make the structure of each configuration clear. The $B = 5$ Skyrmion consists of two parallel down-pointing pentagons attached to two more parallel up-pointing pentagons, so that four sides of a box are formed. The top of the box is formed by adding two squares, and similarly for the bottom of the box, though, of course, the arrangement of joined squares on the top and bottom have a relative rotation of 90°. The $B = 6$ configuration consists of two halves, each of which is formed from a square with a pentagon hanging down from each of its four sides. Note that to join these two halves implies that the two squares are parallel, but one is rotated by 45° relative to the other. The $B = 7$ solid is a regular dodecahedron. The $B = 8$ Skyrmion has a similar structure to its $B = 6$ counterpart, except that the squares are replaced by hexagons, so that each half has six pentagons hanging down. This requires the top hexagon to be parallel to the bottom hexagon but rotated by 30°. Finally, the $B = 9$ structure has four hexagons located at the vertices of a regular tetrahedron which are joined by four sets of three connected pentagons, whose single common vertex lies at the vertices of the dual tetrahedron.

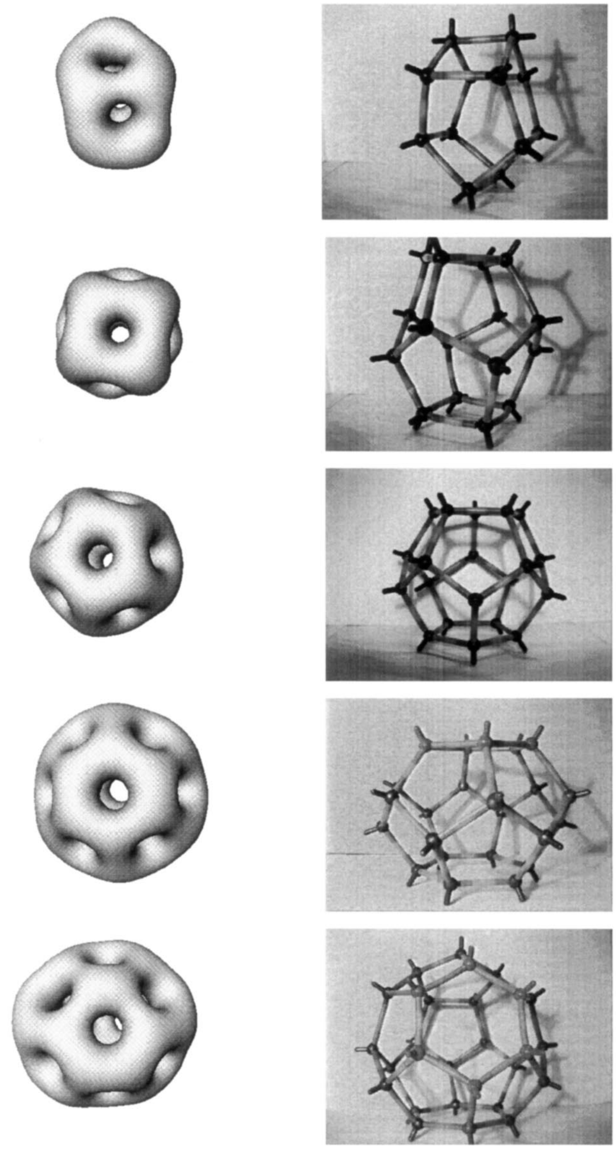

FIG. 1. Skyrmions of charge 5 to 9; on the left baryon density isosurfaces (to scale) with 5 at the top and 9 at the bottom and on the right wire frame models of the corresponding solids. Note that the wire frame models are not to scale and have different orientations to the baryon density plots.

The symmetries of each Skyrmion are of interest, and here we shall discuss the symmetry group of the baryon density (or equivalently energy density). To discuss the symmetries of the field itself is a more complicated task, since the three-dimensional representation of the symmetry group which acts on the pion fields as an isospin rotation must also be identified. In the final column of Table I we give the symmetry group, where we use the Schönflies notation (see, for example, Ref. [9]) popular in chemistry. All the configurations contain at least the symmetry group D_{nd}, for some n. This symmetry group is obtained from the cyclic group of order n, C_n, by the addition of a C_2 symmetry with axis perpendicular to the main symmetry axis and a reflection symmetry in a vertical

TABLE I. The constituent polygons and symmetry structure for the candidate minima of charge 3 and above. T, S, P, and H stand for triangles, squares, pentagons, and hexagons, respectively.

B	Faces	T	S	P	H	Symmetry
3	4	4	0	0	0	T_d
4	6	0	6	0	0	O_h
5	8	0	4	4	0	D_{2d}
6	10	0	2	8	0	D_{4d}
7	12	0	0	12	0	I_h
8	14	0	0	12	2	D_{6d}
9	16	0	0	12	4	T_d

plane containing the main axis and which bisects the angle between pairs of C_2 axes. Thus these twisted dihedral symmetries (which are enhanced to platonic symmetry in some cases) appear to be of importance to Skyrmions, as they are for BPS monopoles [10,11].

Note that the Skyrmions of charges 3, 4, 7 have the same platonic forms as the corresponding Bogomol'nyi-Prasad-Sommerfeld (BPS) monopoles of the same charge [10,12,13], but that the charge 5 configuration is not an octahedron, even though an octahedral 5-monopole exists [10]. An octahedral charge 5 Skyrmion does exist but it is not the minimal energy configuration [7]. This illustrates an important difference between BPS monopoles and Skyrmions. All the monopole configurations of a particular charge have the same energy and it is due to a mathematical simplification that only the very symmetric ones have been found, but the Skyrmions evolve under the influence of a potential allowing for the possibility of minima.

Given the complex structure of these Skyrmions the question now arises as to whether a rule exists which fits the remarkable sequence of shapes found. We propose the following phenomenological rule for the structure of the minimum energy charge $B > 2$ Skyrmion, which we refer to as the geometric energy minimization rule.

GEM rule.—The charge B baryon density surface is composed of almost regular polygons and consists of $4(B-2)$ trivalent vertices. If more than one such solid exists, then select the most spherical.

It should be noted that there are several equivalent ways in which the GEM rule could be stated. For example, from the trivalent property together with Eulers formula, fixing one of the three parameters of the solid, that is, the number of vertices, faces, and edges, determines the other two. Explicitly we have $V = 4(B-2)$, $F = 2(B-1)$, and $E = 6(B-2)$. Thus since the baryon density isosurface has a hole in the center of each face, then the GEM rule implies the observation of Ref. [5] that the isosurface contains $2(B-1)$ holes. As the value of B increases the number of possible configurations satisfying the first part of the GEM rule grows, and hence the need for the second part. For example, at $B = 6$ in addition to the configuration found there is a second possibility that consists of a hexagon with alternating squares and pentagons rising from its edges, and topped by three joined pentagons. This crownlike configuration, which has cyclic C_3 symmetry, satisfies the first part of the GEM rule, but is not very spherical, having a flat bottom and a pointed top. It may be that some other statement, such as a minimization of the standard deviation of edge lengths, or a similar property for edge angles, is an improved statement of the spherical property. More examples of higher charge Skyrmions are required to resolve this issue.

It follows from the GEM rule that for $B \geq 7$ the solid consists of 12 pentagons and $2(B-7)$ hexagons. Such configurations occur in fullerene chemistry [14,15], where we should compare with the carbon structure $C_{4(B-2)}$. In fullerene chemistry avoiding large curvature is important, so the first fullerene is C_{28}, avoiding a fused quartet of pentagons, which has precisely the form of the $B = 9$ Skyrmion. It would be interesting to see if this correspondence continues, since the $B = 17$ Skyrmion should have the C_{60} Buckminsterfullerene structure.

We have calculated the discrete charge and energy for the minima on 100^3 grids, which are displayed in Table II. These energy values are less than the true values since the grid size is finite, but they are nonetheless within 2% accurate. One can make a better estimate of the overall energy by using the ratio $E_{\rm dis}/B_{\rm dis}$ as discussed in Ref. [4]. This can be seen to be exact to 3 decimal places for the $B = 1$ Skyrmion, and we suggest that it will be so for all the others, since the third decimal place has not changed for many (>1000) time steps. (It should be noted that the quoted values for $B = 1$ to $B = 4$ differ very slightly from those presented in Ref. [4]. The current values are the result of further relaxation of the same configurations.) These values have been used to calculate the ionization energy ($I_B = E_1 + E_{B-1} - E_B$), that is, the energy required to remove one Skyrmion, which in general gives an indication of the classical binding energy. We have also calculated the average size of the soliton Δr from the second moment of the baryon distribution as in Ref. [5]. This value is extremely rough since it ignores

TABLE II. Calculated values of the charge ($B_{\rm dis}$), energy ($E_{\rm dis}$), and soliton size (Δr) for $B = 1$ to $B = 9$ in natural units, where the Bogomolyni bound is $E \geq |B|$. Also presented is the ionization energy I, which is the energy required to remove one Skyrmion. For a discussion of the accuracy of the results and comparison to previous calculations see the text.

B	$B_{\rm dis}$	$E_{\rm dis}$	$E_{\rm dis}/B_{\rm dis}$	E	I	Δr
1	0.984	1.212	1.232	1.232		1.034
2	1.972	2.308	1.171	2.342	0.122	1.416
3	2.960	3.384	1.143	3.429	0.145	1.636
4	3.948	4.407	1.116	4.464	0.197	1.860
5	4.935	5.509	1.116	5.580	0.116	2.035
6	5.923	6.567	1.109	6.654	0.158	2.220
7	6.913	7.596	1.099	7.693	0.193	2.332
8	7.900	8.690	1.100	8.800	0.125	2.487
9	8.885	9.759	1.098	9.852	0.150	2.624

the symmetries of the object, but nonetheless it gives an indication of the overall trend.

We should comment on the relevance of our work to previous calculations. In numerical work [5], similar to ours, configurations up to charge 6 were studied using a global minimization algorithm. Our results for charges 5 and 6 differ from these earlier computations, which we attribute to numerical effects in the earlier work due to lack of resources; for example, the grid we use contains almost 5 times the number of points. For charge 5 the difference is small, and we obtain the same symmetry group but identify different polygons forming the solid, which is possible thanks to our improved grid resolution. For charge 6 our results are very different, as we find a very symmetric configuration, whereas the earlier computation gave a structure with very little symmetry.

A different approach to constructing high charge configurations has used the Skyrme crystal [16] and involves cutting out sections [17]. Although these configurations have low energy, they are not as low as those presented here, and they are fundamentally different in nature. The configurations presented here are shells with less baryon density in the center, whereas those created from the Skyrme crystal have internal structure since they are created from cubic configurations. Such structures do not fit the GEM rules since they are not trivalent. It is an open question as to whether the shell structure persists for higher charge.

We believe that the candidate minima which we have presented here are, in fact, the global minima since the initial conditions have natural asymmetry, we have in some cases got the same configuration from two different initial conditions and most of all the isosurfaces of the baryon density fit a remarkable sequence of polygons. The symmetry properties of these polygons could be the starting point for an understanding of the moduli space structure of the Skyrme model, leading to a study of low energy dynamics.

Finally, returning to the application of Skyrmions to nuclear physics, it is encouraging that physical properties, such as classical binding energies, appear to follow the trend of the light elements, at least qualitatively. For example, the binding energy of the $B = 4$ Skyrmion, which models the α particle, is much greater than that of the $B = 5$ Skyrmion, for which there is no corresponding mass number $A = 5$ stable nuclear ground state. Of course, only a rough qualitative correspondence should be expected at this level, since a quantization of the classical soliton solutions is required before a reasonable comparison can be made. However, the classical results are such that one might hope that upon quantization the classical binding energies are reduced, which could lead, for example, to the result that the weakly bound classical $B = 5$ Skyrmion becomes unstable. Note that since we are dealing with a solitonic treatment of particles then we expect that classical results do give important information, in contrast to the more traditional quantum mechanical treatment of nuclei where it is not possible to extract a purely classical picture. Quantization of the classical soliton solutions is a difficult task and remains the aim of a long term investigation, though recent work [18] on the normal mode spectra of the classical $B = 2$ and $B = 4$ Skyrmions is an important step in this direction. A vital ingredient in this analysis is the symmetry of the Skyrmion, so our results on the very symmetric higher charge Skyrmions is the first necessary ingredient in extending these results to $B > 4$ Skyrmions. It is interesting to note that recently it has been discovered [19] that realistic nuclear force models actually do predict highly anisotropic density distributions in light nuclei, with little density at the origin.

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