Can Wave Packet Revivals Occur in Chaotic Quantum Systems?

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The short time revivals of initially localized wave packets are well known in simple, closed, 1-degree-of-freedom (1D) systems. In 2D or higher, if the system is integrable or has exclusively periodic dynamics, a generalization is possible. If the dynamics are chaotic, revivals have not been previously seen and are, *a priori*, not expected. Nevertheless, we have found that some stretched wave packets in a chaotic system experience, very early, surprisingly large recurrences. We extend a semiclassical theory founded on summing over heteroclinic orbits to determine a set of necessary conditions. The most important one is an Einstein-Brillouin-Keller-like quantization of classical flux crossing the turnstile formed by the stable and unstable manifolds of the initial wave packet's underlying central orbit. [S0031-9007(97)04472-4]

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In most quantum systems, initially localized wave packets will spread and disperse as they are propagated. For systems with classical analogs exhibiting periodic dynamics, such wave packets will reconstruct at relatively short times leading to "revivals" of the initial localized state [1]. The most important and recent physical examples are given by coherent electronic wave packets in Rydberg atoms which have been theoretically treated [1] and experimentally measured [2]. The coulomb problem is not unique though, and, quite generally, bounded 1-degree-of-freedom (1D) systems possess periodic dynamics, and thus manifest revival behavior. In the short wavelength limit, their spectra are locally uniform (harmonic-oscillator-like) except that the energy spacing between levels is slowly and smoothly changing. Both properties are essential for the initial dispersion and subsequent revivals. An alternative semiclassical approach makes direct use of the underlying classical evolution to explain the main quantum features including fractional revivals [1,3].

In systems with more than 1D, the classical dynamics is almost always quasiperiodic (integrable), chaotic, or some mixture of both. Early revivals of fully localized wave packets can no longer be expected. For 2D integrable systems, one way to circumvent this problem is to create a stretched wave packet which is only localized transverse to some periodic orbit, but carrying a phase and slowly varying amplitude of the form $A \exp(i \int p \cdot dq/\hbar)$ along the orbit. This effectively reduces the wave packet's underlying dynamics to 1D. In this manner, the stretched wave packet is constructed from a superposition of a portion of the eigenstates whose energy level spacings are nearly uniform. This cannot be done for chaotic systems whose energy levels repel as in random matrix theory and whose eigenstates are not localized to tori in phase space via Einstein-Brillouin-Keller (EBK) quantization [4]. Nevertheless, in this Letter we show examples of stretched wave packets in the stadium billiard, a paradigm of chaos, with behavior similar to the usual revival behavior found in periodic dynamical systems. Although the occurrences are rare and only at particular wavelengths, it is remarkable that any chaotic revivals exist. We then show that the semiclassical theory of wave packet propagation in chaotic systems [5] can be further developed for stretched wave packets with little modification. From this we determine general conditions necessary to observe "chaotic revivals" and relate their behavior to geometric phase space properties. More specifically, the central, underlying, classical trajectory's stable and unstable manifolds cross to form a complicated "broken separatrix" which encloses a phase space volume and has an associated turnstile determining the classical flux of trajectories crossing in and out of this volume [6]. The first two criteria are EBK-like quantization conditions on the phase space volume inside and, much more importantly, the flux crossing through the turnstile.

From a dynamical viewpoint it is straightforward to see why revivals in chaotic systems should not occur; we restrict ourselves to 2D, bounded systems in a semiclassical regime. Chaotic systems being unstable, the wave packet will rapidly disperse in the transverse degree of freedom. Once dispersed, the underlying trajectories repeatedly explore the available phase space. At any given moment in time, between the initial central phase point's neighborhood and any final point's neighborhood, the wave function will be constructed with many groups of trajectories which have followed a large number of random-looking paths. All those contributions ending at a particular position will interfere to give the total wave function at that point. Assuming the accumulated phases acquired along each path are more or less random, the evolution must appear as random moving waves; see Fig. 7 of [7]. This randomlike evolution would continue for enormously long times, far beyond the time scales under consideration, after which the discreteness of the spectrum eventually enforces recurrences.

Typically, this image is borne out. In the left column of Fig. 1 we show time snap shots of the evolution of the initial state constructed along the horizontal periodic orbit of the stadium billiard pictured in the uppermost frame. It rapidly disperses and then oscillates in some complicated way with little discernible pattern. However, for some specific values of the wave vector, the evolution has a very different appearance. In the right column of Fig. 1, we see a case in which the initial state relocalizes almost completely at just beyond double the period of its underlying periodic orbit. Furthermore, it continues to relocalize repeatedly.

This unusual behavior is not predicted by Heller's original linearized wave packet dynamics applied to scarring of eigenstates [8], nor has it anything to do with localization due to vertical bouncing ball motion. With respect to the former, the instability is too great to predict such large recurrences, and with respect to the latter, the revivals occur so early in time that the slow entrance of the dynamics into the bouncing-ball phase space domain is not yet apparent. Neither is a spectral argument possible in this case as for integrable systems. This is for the same reasons that revivals are not expected in chaotic systems. To study the revivals more quantitatively, we consider the initial state's ($|\alpha\rangle$) autocorrelation function $C_\alpha(t) = \langle\alpha|\alpha(t)\rangle$. Initially, $C_\alpha(0)$ is unity from normalization and its decay depends on the instability of the orbit, but is generally rapid. We consider the system as exhibiting a chaotic revival when $C_\alpha(t)$ returns close to unity at a time beyond which the initial state has completely dispersed, but long before the quantum recurrence implied by the discreteness of the spectrum. In Fig. 2, we show $C_\alpha(t)$ for the two stretched wave packets of Fig. 1. The nonreviving wave packet displays a complicated, fluctuating time dependence. It is not shown here, but this behavior is very similar to that found with Gaussian orthogonal ensemble (GOE) simulations. To make the GOE comparison it suffices to fix an effective dimensionality given by the energy uncertainty determined from the initial decay rate of $C_\alpha(t)$ divided by the local level density of the stadium. In stark contrast, one can easily see the reviving wave packet evolves in a very surprising, nearly periodic fashion up to and far beyond the Heisenberg time scale defined by the mean level spacing. Its initial dispersion shows no hint of the impending, nearly complete rebuilding of the waves into the initial state.

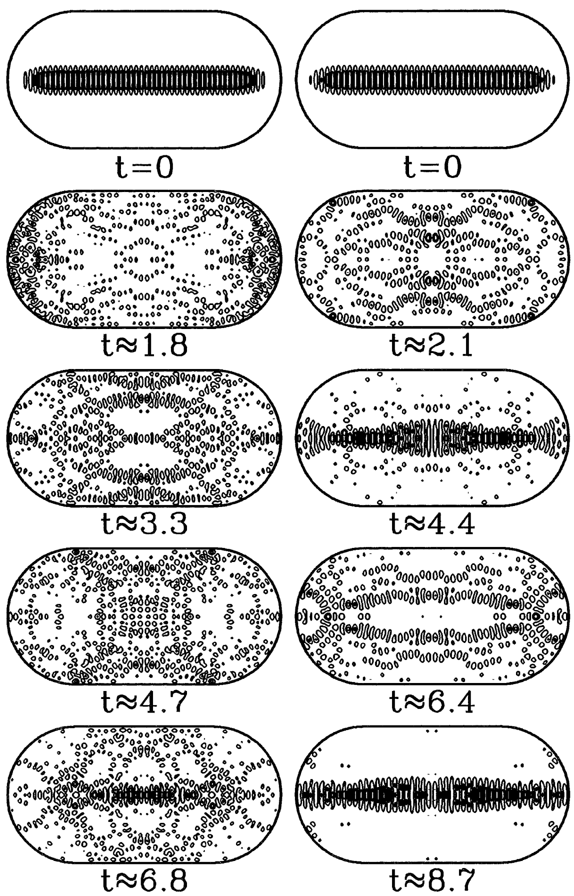

FIG. 1. The time evolution of two stretched wave packets in the Bunimovich stadium. The evolution of an initial wave packet spanning 27 wavelengths is shown in the left column. On the right, the wave packet is 24 wavelengths across. Two contour levels are drawn for $|\psi_\alpha(t)|^2$. Note that $t = 1$ is the time to cross the stadium horizontally at the mean momentum.

Turning to a theoretical explanation of the revival, if $|\gamma\rangle, \langle\beta|$ are localized in all degrees of freedom, then in Ref. [5] it was shown that

$$C_{\beta\gamma}(t) \equiv \langle\beta|\gamma(t)\rangle \approx \sum_j A_j(t) e^{iS_j(t)/\hbar - i\nu_j\pi/2}, \quad (1)$$

where the index j runs over all orbits heteroclinic to the central trajectory of $|\gamma\rangle$. A_j is a slowly varying complex envelope, S_j is a classical action which changes with time, and ν_j is a phrase index. The role of the heteroclinic orbits is to provide a natural scheme for organizing all the returning dynamics at any fixed time.

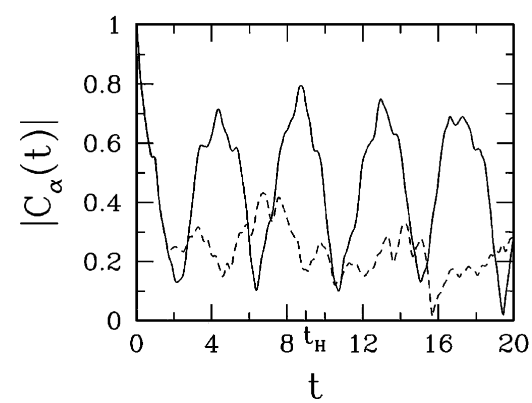

FIG. 2. The quantum autocorrelation function $C_\alpha(t)$. The solid curve corresponds to the right column stretched state of Fig. 1 and the dashed curve corresponds to the state in the left column.

We first decomposed the initial stretched state into a series of Gaussian wave packets displaced along the periodic orbit. Consequently, the autocorrelation function for the stretched state $C_\alpha(t)$ involves a double sum of $C_{\beta\gamma}(t)$ over all pairs of Gaussian wave packets that make up the initial and final stretched states. Note that because the state is stretched along a periodic orbit, essentially identical homoclinic orbits arise for each pair. In fact, we are using finite segments of the infinitely long homoclinic orbits and it is only the segment endpoints which change from term to term; here we use the term orbits for segments without distinguishing between the two. It turns out that first performing the double summation gives a $C_\alpha(t)$ which has the form of Eq. (1) except with a modified amplitude and phase for each orbit. The contributions to $C_\alpha(t)$ from two primary homoclinic orbits are shown in Fig. 3. As can be seen in this figure, there is a window of time, roughly the period of the periodic orbit, say τ, during which a given homoclinic orbit contributes to $C_\alpha(t)$. The crucial question is then, "how do we understand the interference properties of homoclinic orbit contributions whose periods fall within τ of each other?"

We focus on the relative phases of different homoclinic orbit contributions. In Fig. 3, it can be seen that over the time window in which two orbits contribute and are both significant, the phase difference between them is, for all practical purposes, a constant. The constancy is a continuous time manifestation of an asymptotic simplicity given by Eq. (A3) in the Appendix of O'Connor *et al.* [9]. Once the eigenvalue associated with the unstable manifold λ and the stable manifold $1/\lambda$ satisfy the condition $\lambda \gg 1/\lambda$, the expressions reduce to factors that depend on the initial and final states multiplied by an amplitude and phase that involve only the orbit, but is independent of the states.

This result represents a great simplicity since it renders the main problem one of systematically studying the relative homoclinic orbit action differences on a fixed energy surface. Using known techniques [6], it is possible to relate them to phase space structures. The main idea is that the homoclinic orbits lie at the intersections of the stable and unstable manifolds which are also Lagrangian invariant manifolds. In such situations, to calculate action differences it is permissible to deform paths (i.e., orbits) along either manifold as long as the endpoints remain fixed. In Fig. 4, we show the simplest example. There are two primary homoclinic orbits; see Fig. 3. First, follow the emboldened path along the manifolds from the periodic point C to the homoclinic point 2 and back to the periodic point C'. The area enclosed by the path and the $p = 0$ axis (resonance zone) is the difference in action between the guiding periodic orbit and the shortest primary homoclinic orbit. On the other hand, if we follow $C \to 1 \to A \to C'$, then one has the difference with respect to the second primary homoclinic orbit. Comparing the two, one path encloses an additional loop of the turnstile pictured with respect to the other. Thus, the turnstile flux is the action difference between the two primary homoclinic orbits, and it controls the relative phase contribution among entire families of homoclinic orbit segments.

Beyond the primaries are an infinite number of other families of orbits. More and more of them contribute as time increases. In fact, in Fig. 2, approximately 50 independent homoclinic paths are important in reconstructing the first revival at $t \approx 4.35$. Each revival afterwards involves 1000 times more orbits than the previous one. The reason that chaotic revivals are so unexpected is that it is difficult to imagine how wave

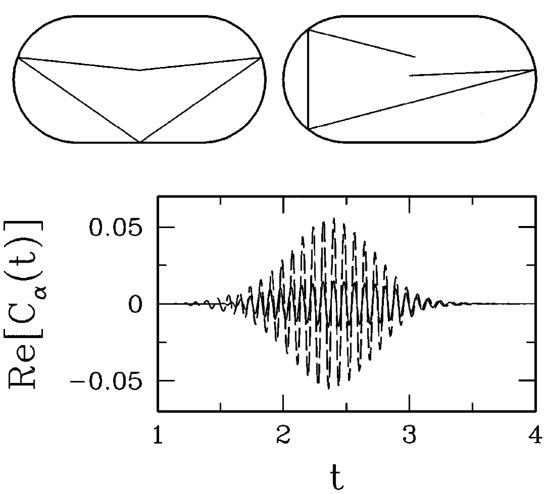

FIG. 3. Contributions to the autocorrelation function from the two primary homoclinic orbits peictured above. The solid curve corresponds to the upper right orbit and the dashed curve to tthe upper left. At the same wavelength as the chaotic revival of Fig. 1, the semiclassical amplitudes of the homoclinic primaries constructively interfere.

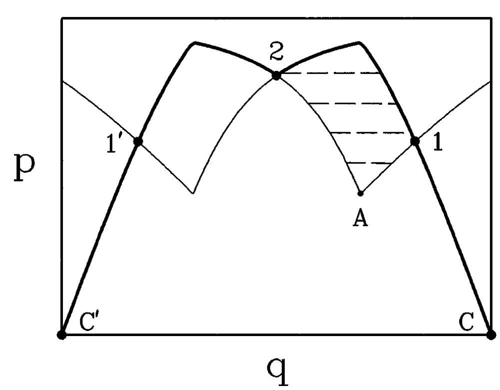

FIG. 4. The stable and unstable manifolds of the horizontal bouncing periodic orbit. Only one-quarter of the full phase plane is shown. The unstable manifold originates from point C and the stable manifold from C'. Points 1 and 1′ correspond to the upper right primary homoclinic orbit of Fig. 3 and point 2 the upper left. The dashed area corresponds to the flux exiting the resonance zone.

amplitude following seemingly random paths can lead to a rebuilding of the initial state. Subtle, weak correlations among orbit actions are actually necessary. The same construction method applies for these other orbit families, though the action differences have more complicated expressions; see [10] for a full discussion. The key result is that from a fairly small number of areas (actions), it is possible to generate an enormous number of orbit action differences leading necessarily to correlations among the orbit actions. To the extent that quantizing or nearly quantizing certain areas is possible, significant constructive phase interference is guaranteed between the primaries and also others. In fact, we used this quantization to locate the example shown. Because the chaotic revivals are rare and difficult to find, randomly choosing some wave vector and testing for whether a revival existed turned out to be too inefficient to be of any help.

Quantizing the resonance zone, including the phase index, enforces constructive interference between the periodic orbit and a single primary homoclinic orbit family. By quantizing the turnstile both primary families are constructively interfering and action differences depending on multiples of the turnstile action must return in phase as well. In the stadium, it turns out that more wave amplitude is controlled by the homoclinic primaries than the guiding periodic orbit at the wavelengths considered. Thus, the most important criterion for finding revivals in the stadium in this wavelength regime turned out to be the quantization of the turnstile. However, note that we could alter the sidelength to values for which both resonance and turnstile areas would quantize simultaneously (which is not the case in Fig. 1) leading to more perfect revivals than the example shown in the Letter [11].

As $\hbar \rightarrow 0$, quantizing only the turnstile will become insufficient to imply a chaotic revival since less and less amplitude at short times will be controlled by the primary families. New quantization conditions will arise (an infinity of them) depending on how the manifolds slice through the turnstile and on action corrections arising from piecing together complicated orbits from the primaries. It is unknown to us whether a sufficient number of conditions will be able to be simultaneously fulfilled to continue to imply revivals for special $\hbar$ values.

The arguments given here are general and do not depend on any special property of billiard systems. They require only that the manifolds are continuous. One physical example of a system to which our logic applies is the diamagnetic hydrogen atom [12]. It may be possible therefore to find similar behavior there.

To conclude briefly, we have found some examples of stretched wave packets which nearly revive in a chaotic system. Their behavior exhibits similarities and differences with respect to their counterparts in integrable systems. We have given some necessary conditions for their existence, but more work is needed to know what is both necessary and sufficient, especially as $\hbar \rightarrow 0$. An analysis of the semiclassical theory shows that the key determination of interference patterns is from the action differences of returning orbits which can be organized by the homoclinic orbits. Furthermore, there are action correlations in these returning paths which can be accounted for in this context. With these conditions, it is possible to search systematically for examples of chaotic revivals which are rare. Also it is predicted that tuning a parameter such as the sidelength will lead to special systems in which the revivals are even more spectacular.

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