Domain Wall Quarks and Kaon Weak Matrix Elements

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We present lattice calculations of kaon matrix elements with domain wall fermions. Using lattices with $6/g^2 = 5.85$, 6.0, and 6.3, we estimate $B_K(\mu \approx 2 \text{ GeV}) = 0.628(47)$ in quenched QCD, which is consistent with previous calculations. At $6/g^2 = 6.0$ and 5.85 we find the ratio f_K/m_ρ, in agreement with the experimental value, within errors. These results support expectations that $O(a)$ errors are exponentially suppressed in low energy ($E \ll a^{-1}$) observables, and indicate that domain wall fermions have good scaling behavior at relatively strong couplings. We also demonstrate that the axial current numerically satisfies the lattice analog of the usual continuum axial Ward identity.
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While lattice gauge theory has made significant progress in addressing the outstanding challenge of calculating hadronic observables from first principles, a basic feature of the strong interactions has been missing in these calculations, the $SU(N_f)_L \times SU(N_f)_R$ chiral flavor symmetry of the light quarks which is broken explicitly by present lattice discretizations of continuum QCD. We recently reported [1] on calculations using a new discretization for simulations of QCD, domain wall fermions (DWF) [2,3], which preserve chiral symmetry on the lattice in the limit of an infinite extra 5th dimension. There it was demonstrated that DWF exhibit remarkable chiral behavior [1] even at relatively large lattice spacing and modest extent of the fifth dimension. Here we give further results using DWF which are of direct phenomenological interest.

In addition to retaining chiral symmetry, DWF are also "improved" in another important way. In the limit that the number of sites in the extra dimension N_s goes to infinity, the leading discretization error in the effective four dimensional action for the light degrees of freedom goes like $O(a^2)$, unlike the case for ordinary Wilson fermions, for which the errors are $O(a)$, a being the lattice spacing. This theoretical dependence is deduced from the fact that the only operators available to cancel $O(a)$ errors in the effective action are not chirally symmetric; thus no $O(a)$ errors exist in the low energy theory. For finite N_s, $O(a)$ corrections are expected to be exponentially suppressed with the size of the extra fifth dimension. Our calculations for B_K show a weak dependence on a that is easily fit to an a^2 ansatz. At $\beta = 6.0$ ($\beta \equiv 6/g^2$) the lattice spacings determined from m_ρ and f_π agree within less than five percent. This improved scaling behavior is plausible in light of the fact that DWF retain an important continuum symmetry at nonzero lattice spacing.

As in our previous paper, we use the boundary fermion variant of DWF [2] developed by Shamir [3]. The DWF action is essentially a five dimensional analog of the ordinary Wilson fermion action with two key differences: (1) the relative sign between the Wilson term and the (five dimensional) Dirac mass M is opposite to the usual

convention. This leads to the appearance of massless chiral modes on the boundaries of the fifth dimension, a left-handed fermion on one wall and a right-handed one on the other. (2) The layers $s = 0$ and $s = N_s - 1$ (s denotes the coordinate in the extra dimension) are coupled with strength $-m$ ($m \gtrsim 0$). Neglecting exponentially small corrections, in Ref. [3] it was shown that the parameter m is (proportional to) the mass of the light four dimensional quark which is assembled from the two chiral modes $m_q = mM(2 - M)$. The chiral limit is $N_s \to \infty$ and $m \to 0$, which requires no fine tuning unlike ordinary Wilson fermions.

Recently, the exponentially small corrections to the quark mass have been given at tree level [4], $m_q = M(2 - M)\{m + [1 - M + O(p^2)]^{N_s}\}$. In the presence of interactions M is renormalized additively just like ordinary Wilson fermions which also acquire a "mass term" proportional to p^2. Perturbatively at one loop the main effect of the interactions is the replacement $M \to M + g^2 \Sigma(p)$ [3] where $\Sigma(p)$ is the quark self-energy. Thus the chiral limit still holds with the replacement $(1 - M)^{N_s} \to (M_c - M)^{N_s}$. This is analogous to the renormalization of the critical hopping parameter from its tree level value of $1/8$ in the case of ordinary Wilson fermions. Of course, the crucial difference is that for DWF the additive corrections are exponentially suppressed. In our original study we found $M_c \approx 1.7$ for nonperturbative couplings corresponding to quenched simulations at $\beta \sim 6.0$, which also agrees roughly with a simple mean field argument [1].

For QCD, the DWF are gauged in the ordinary four dimensions only, and the left and right-handed modes couple equally to the gauge field. Thus the five dimensional theory gives rise to a low energy effective theory ($E \ll a^{-1}$) describing interacting vector quarks in four dimensions whose right and left-handed components are localized around $s = 0$ and $s = N_s - 1$, respectively.

In Ref. [5] it was shown that operators constructed from the quark fields formed by the chiral modes on each wall satisfy the following four dimensional chiral Ward

identities (CWI):

$$\Delta_\mu \langle A_\mu^a(x) O(y_1, y_2, \ldots) \rangle = 2m \langle J_5^a(x) O(y_1, y_2, \ldots) \rangle$$
$$+ 2 \langle J_{5q}^a(x) O(y_1, y_2, \ldots) \rangle$$
$$+ i \langle \delta_A^a O(y_1, y_2, \ldots) \rangle, \quad (1)$$

which result from demanding invariance of $\langle O(y_1, y_2, \ldots) \rangle$ under an infinitesimal axial transformation δ_A^a. Here A_μ^a is the four dimensional partially conserved axial current (PCAC) constructed from a sum of five dimensional fields over all slices in the extra dimension [5]. The operators $O(y_1, y_2, \ldots)$ and the pseudoscalar density $J_5^a(x)$ are constructed from four dimensional quark fields using the chiral modes on each boundary,

$$q(x) = \frac{1 + \gamma_5}{2} \psi(x, 0) + \frac{1 - \gamma_5}{2} \psi(x, N_s - 1),$$
$$\overline{q}(x) = \overline{\psi}(x, N_s - 1) \frac{1 + \gamma_5}{2} + \overline{\psi}(x, 0) \frac{1 - \gamma_5}{2}. \quad (2)$$

In Eq. (1), J_{5q}^a is an anomalous pseudoscalar density that results from the noninvariance of the action under the axial transformation for finite N_s. For flavor nonsinglet currents, this contribution to the right-hand side (rhs) of Eq. (1) vanishes identically in the limit $N_s \to \infty$ [5], and we are left with the continuum-like relations. Below we demonstrate explicitly that at $\beta = 6.0$ Eq. (1) is satisfied for the usual PCAC Ward identity $O(y_1, y_2, \ldots) = J_5$, and the anomalous contribution is small for $N_s = 10$ and reduces further by more than a factor of 2.5 as N_s is increased to 14.

To obtain B_K we need the matrix element of the $\Delta s = 2$ four quark operator that governs $K - \overline{K}$ mixing, $O(\mu)_{LL} = [\overline{s} \gamma_\nu (1 - \gamma_5) d]^2$, which depends on the energy scale μ. On the lattice and using DWF, O_{LL} is a simple transcription of the above using the quark fields in Eq. (2). Sandwiching O_{LL} between K and $\overline{K}$ states and taking the ratio with its value in vacuum saturation yields B_K.

In order to investigate the continuum limit of quenched QCD with DWF, we have carried out simulations at gauge couplings $\beta = 5.85, 6.0,$ and 6.3. The simulation parameters are summarized in Table I. The number of configurations in our study is rather small, and we have made no attempt to estimate finite (four dimensional) volume systematic errors. These deficiencies will, of course, be addressed in future works. The lattices correspond to $(1.5 \text{ fm})^3$ for $\beta = 6.0$ and 6.3 and $(2.1 \text{ fm}^2)^3$ for 5.85,

TABLE I. Summary of simulation parameters. "Size" is the number of spatial sites times the temporal extent times N_s. M is the five dimensional Dirac fermion mass, and m is the coupling between layers $s = 0$ and $N_s - 1$. The number in parentheses is the number of configurations (# conf) used at each value of m.

β	Size	M	m (# conf)
5.85	$16^3 \times 32 \times 14$	1.7	0.075(34), 0.05(24)
6.0	$16^3 \times 32 \times 10$	1.7	0.075(36), 0.05(39), 0.025(17)
6.3	$24^3 \times 60 \times 10$	1.5	0.075(11), 0.05(14)

which from previous lattice studies may result in a deviation of a few percent from the infinite volume case. We have begun to address systematic corrections due to finite N_s. All of the correlation functions discussed below were calculated in the lattice Landau gauge which was chosen for convenience, in principle any gauge will do.

We begin by discussing the numerical investigation of Eq. (1) for $O = J_5$. First, Eq. (1) is satisfied exactly on any configuration since it is derived from the corresponding operator identity. We checked this explicitly in our simulations. In the asymptotic large time limit, we get

$$2 \sinh(am_\pi/2) \langle A_\mu | \pi \rangle / \langle J_5 | \pi \rangle$$
$$= 2m + 2 \langle J_{5q} | \pi \rangle / \langle J_5 | \pi \rangle, \quad (3)$$

which goes over to the continuum relation for $am_\pi \ll 1$ and $N_s \to \infty$. At $\beta = 6.0$ and $N_s = 10$ we find the left-hand side (lhs) of Eq. (3) to be 0.1578(2) and 0.1083(3) for $m = 0.075$ and 0.05, respectively. The anomalous contributions for these two masses are $2 \times [0.00385(5)$ and 0.00408(12)]$, which appears to be roughly constant with m. Increasing N_s to 14 at $m = 0.05$, the anomalous contribution falls to $(2\times) 0.00152(8)$ while the lhs is 0.1026(6), which shows that increasing N_s really does take us towards the chiral limit.

While we have not investigated the CWI for O_{LL}, the matrix element $\langle K^0 | O_{LL} | \overline{K}^0 \rangle$ vanishes linearly with m in the chiral limit as required by chiral perturbation theory and shown in Fig. 1. This indicates that the anomalous term in Eq. (1) for O_{LL} is highly suppressed. At $\beta = 5.85$ the two data points extrapolate linearly to $-0.0005(100)$ at $m = 0$. At $\beta = 6.0$ the three data points extrapolate to $-0.004(9)$ with a χ^2 per degree of freedom of 0.2 for a noncovariant fit. We do not have enough data to perform a covariant fit. At $\beta = 6.3$, the two points extrapolate to 0.05(3). This slight overshoot is not unexpected since the values of m used rather
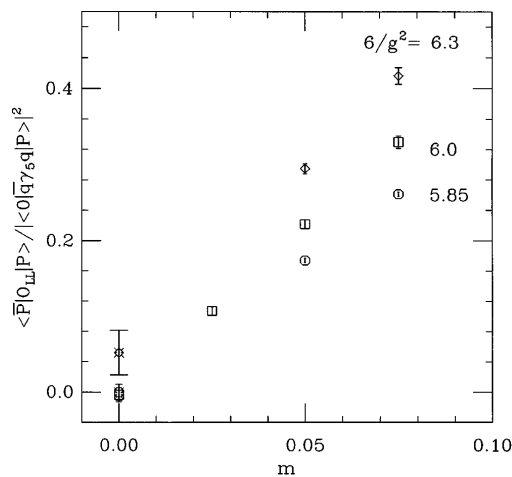

FIG. 1. The matrix element $\langle \overline{P} | O_{LL} | P \rangle$ vs m. $|P\rangle$ is a nonsinglet pseudoscalar state. m is proportional to the quark mass in lattice units. At $\beta = 5.85$ and 6.0 $\langle \overline{P} | O_{LL} | P \rangle$ vanishes linearly with m. The slight overshoot at $\beta = 6.3$ is likely due to higher order terms in the chiral expansion (see text).

heavy quarks. In our initial study we found a similar behavior [1], and as the quark mass was lowered, the required linear behavior set in. Since we do not have a smaller mass at this coupling, there will be a small systematic increase in B_K since the fit will overestimate the matrix element at the strange quark mass. All of the above results are for $N_s = 10$ except at $\beta = 5.85$, where $N_s = 14$ was used for reasons explained below.

Figure 2 shows B_K as a function of am_ρ, or equivalently the lattice spacing. B_K is estimated at each coupling from a linear fit of the degenerate quark data. The fit is then evaluated at one-half the value of m corresponding to the strange quark as determined from a fit to the pseudoscalar mass squared (see below). am_ρ is determined from a simple jackknife average of the effective mass over a suitable plateau. The results for B_K depend weakly on β, and are well fit to a pure quadratic in a. Using this fit, we find $B_K(\mu = a^{-1}) = 0.628(47)$ in the continuum limit with a confidence level of 0.39. This result is already consistent with the previous Kogut-Susskind result [6,7] and a recent Wilson quark result [8] using CWI's similar to Eq. (1) to enforce the proper chiral behavior of O_{LL} for Wilson quarks. We note that the data can be fit to a linear function of the lattice spacing as well, which yields $B_K(\mu = a^{-1}) = 0.617(80)$, though we emphasize again that linear corrections are expected to be highly suppressed on theoretical grounds. More importantly, there is no evidence for $O(a)$ corrections in Fig. 1. Similarly, the denominator in B_K, $\langle \overline{K}|O_{LL}|K\rangle_{VS}$, exhibits the correct chiral behavior. Above, the notation $B_K(\mu = a^{-1})$ simply means the uncorrected lattice data have been used to perform the extrapolation; i.e., our result does not include the perturbative running of B_K at each lattice spacing to a common energy scale. This requires
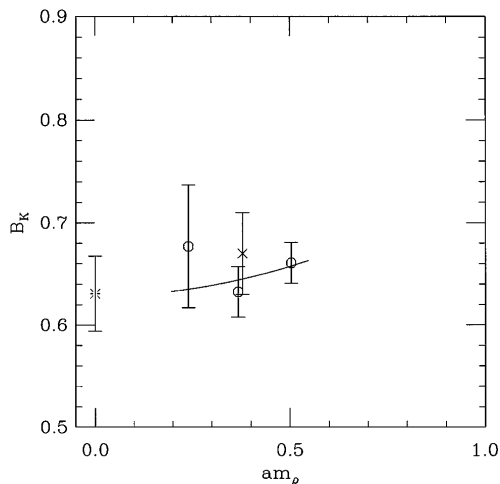

FIG. 2. Kaon B parameter. The solid line is a pure quadratic fit to the data, and the burst denotes the extrapolation to the continuum limit $a = 0$. The data are for $N_s = 10$ ($\beta = 6.0, 6.3$) and 14 (5.85). The cross (not used in the fit) denotes the partially unquenched result discussed in the text. The energy scale is roughly 2 GeV at $\beta = 6.0$, and perturbative corrections have not been included.

a perturbative calculation to determine the renormalization of O_{LL}, which has not yet been done. The energy scale at $\beta = 6.0$, as determined by the inverse lattice spacing, is roughly 2 GeV. Since the dependence on a is already mild, this should not have a significant impact on our $a = 0$ estimate. We also note that the coefficient of the quadratic term in our fit is 0.12(23), or zero within errors. Also, from our previous work [1] which did not give a value for B_K, we find $B_K(\mu = a^{-1}) = 0.67(4)$ on a set of 20 Kogut-Susskind lattices with $m_{KS} = 0.01$ and $\beta = 5.7$ [9] and the same five dimensional lattice volume as the point at $\beta = 6.0$. The energy scale is nearly that of $\beta = 6.0$, quenched. This partially unquenched result indicates that the error from quenching may be small, as was found for B_K using Kogut-Susskind quarks [10,11].

At $\beta = 6.0$, we have also calculated B_K using the partially conserved axial current $A_\mu^a(x)$ (and the analogous vector current) at $m = 0.05$ and 0.075. This point split conserved current requires explicit factors of the gauge links to be gauge invariant. Alternatively, a gauge noninvariant operator may be defined by omitting the links; the two definitions become equivalent in the continuum limit. Results for the gauge noninvariant operators agree within small statistical errors with those for operators constructed from Eq. (2). The results for the gauge invariant operators are somewhat larger: $B_K^{\mathrm{inv}}(\mu = a^{-1}) = 0.872(22)$ and 0.926(19) at $m = 0.05$ and 0.075, respectively. A similar situation holds in the Kogut-Susskind case where it was shown that the gauge invariant operators receive appreciable perturbative corrections which bring the two results into agreement [10].

Using Eq. (3), neglecting the anomalous contribution, and using the definition of the decay constant $f_P m_P \equiv \langle 0|A_0^a|P\rangle$, we can determine the pseudoscalar decay constant from the measurement of $\langle 0|J_5^a|P\rangle$. Performing simultaneous covariant fits to the wall-point and wall-wall correlators of J_5 yields the matrix element. At $\beta = 5.85$ and 6.0 each fit has a good confidence level (CL ≈ 0.7). The results are shown in Fig. 3. Proceeding as before with B_K, we find $f_K = 159(14)$ MeV and 164(12) MeV for $\beta = 6.0$ and 5.85, respectively. The errors are statistical and do not include the error in the lattice spacing determination from am_ρ. The central values agree with experiment, $f_{K^+} = 160$ MeV. The lattice spacing determinations from am_ρ give $a^{-1} = 1.53(27)$, 2.09(21), and 3.20(81) GeV at $\beta = 5.85$, 6.0, and 6.3, respectively. These are similar to Wilson and Kogut-Susskind lattice spacings for similar quenched lattices. Alternatively, we may form the dimensionless ratio f_K/m_ρ. We find for $\beta = 5.85$ and 6.0, $f_K/m_\rho = 0.213(42)$ and 0.206(27), where we have added the statistical errors naively in quadrature. The experimental result is 0.208. At present our data at $\beta = 6.3$ are too noisy after extrapolation to give a significant result. Finally, we note that we have also calculated the decay constant directly from the matrix element of the partially conserved axial current at the points $m = 0.05$ and 0.075 at $\beta = 6.0$, and the results agree with
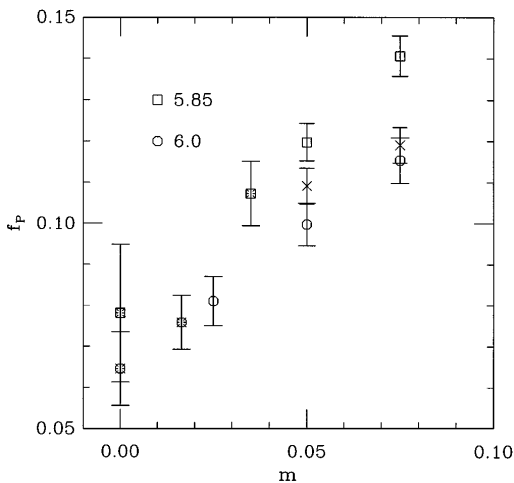
FIG. 3. Pseudoscalar decay constant. Bursts are linear extrapolations to values of m corresponding to the pion and the kaon. $N_s = 10$ and 14 for $\beta = 6.0$ and 5.85, respectively. The crosses denote values calculated from the matrix element of the partially conserved axial current ($\beta = 6.0$ only).
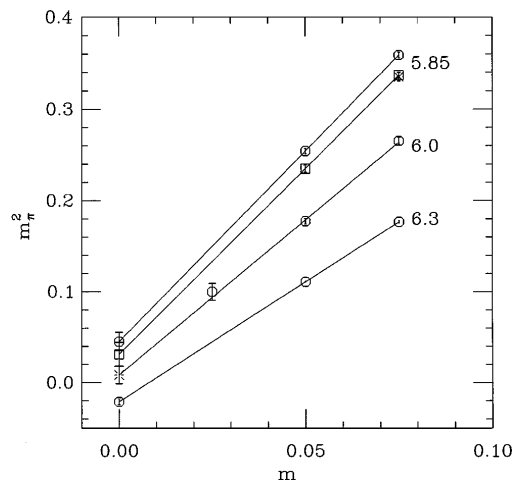

FIG. 4. Pion mass squared. The curve at $\beta = 6.0$ [$N_s = 10$ (octagons)] agrees with the expectation from lowest order chiral perturbation theory. The others show small but significant deviations. At $\beta = 5.85$ going from $N_s = 10$ to 14 reduces the discrepancy. However, increasing N_s to 18 (cross) has no effect. At $\beta = 6.3$, the discrepancy is most likely due to higher order terms in the chiral expansion (see text).

those using the matrix element of the pseudoscalar density (see Fig. 3). While the above results indicate good scaling behavior, they must be checked further with improved statistics and a fully covariant fitting procedure. More importantly, the continuum limit still has to be taken: a recent precise calculation using quenched Wilson quarks by the CP-PACS collaboration gives a value for f_K/m_ρ in the continuum limit that is inconsistent with experiment [12], so the above agreement with experiment may be fortuitous.

In Fig. 4 we show the pion mass squared as a function of m. Lowest order chiral perturbation theory requires m_π^2 to vanish linearly with m. At $\beta = 6.0$ our three data points extrapolate linearly to 0.008(10), which agrees with the above expectations. At $\beta = 6.3$, the two data points extrapolate to $-0.021(3)$, which is once again likely due to quark masses that are too heavy to agree with lowest order chiral perturbation theory. At $\beta = 5.85$ the two masses extrapolate to 0.045(10) for $N_s = 10$ and 0.031(13) for $N_s = 14$. This discrepancy is probably not due to higher order terms in the chiral expansion since the physical quark masses are light compared to the masses at the other couplings, and the curvature would have the wrong sign. We see a large downward shift in m_π^2 as N_s goes from 10 to 14. However, increasing N_s to 18 at $m = 0.075$ has a negligible effect. This behavior may signal a strong coupling effect where the suppression of explicit chiral symmetry breaking terms with N_s may be weakened. In the case of the vector Schwinger model, it was found that topology changing gauge configurations can induce significant explicit chiral symmetry breaking effects [4]. Further investigation is required.

Our study shows that DWF are an attractive alternative to Kogut-Susskind and Wilson quarks for lattice QCD calculations where chiral symmetry is crucial. For weak matrix elements in particular, DWF yield good agreement

with expectations from chiral perturbation theory without the complicated mixing of operators required with Wilson quarks, or the entanglement of flavor and space-time degrees of freedom as with Kogut-Susskind quarks. Perhaps even more importantly, up to exponentially small corrections, DWF maintain the full chiral symmetry of QCD at relatively strong couplings, and thus should have more continuumlike behavior. The data presented here seem to indicate just that, though future studies with improved statistics are needed to confirm this. This improved scaling may compensate for the added cost of the extra dimension.

Our domain wall fermion code relies heavily on the four dimensional MILC code [13], which we are happy to acknowledge again. The numerical computations were carried out on the NERSC T3E. This research was supported by U.S. DOE Grant No. DE-AC0276CH0016.

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