Why Is the Matrix Model Correct?

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We consider the compactification of M theory on a lightlike circle as a limit of a compactification on a small spatial circle boosted by a large amount. Assuming that the compactification on a small spatial circle is weakly coupled type-IIA theory, we derive Susskind's conjecture that M theory compactified on a lightlike circle is given by the finite N version of the matrix model of Banks, Fischler, Shenker, and Susskind. This point of view provides a uniform derivation of the matrix model for M theory compactified on a transverse torus T^p for $p = 0, \dots, 5$ and clarifies the difficulties for larger values of p. [S0031-9007(97)04677-2]

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About a year ago Banks, Fischler, Shenker, and Susskind (BFSS) [1] proposed an amazingly simple conjecture relating M theory in the infinite momentum frame to a certain quantum mechanical system. The extension to compactifications on tori T^p for $p = 1, \dots, 5$ was worked out in [1–5]. This proposal was based on the compactification of M theory on a spatial circle of radius R_s in a sector with momentum $P = N/R_s$ around that circle. In the limit of small R_s, M theory becomes the type-IIA string theory and the lowest excitations in the sector with momentum P are N D0-branes [6]. When the D0-brane velocities are small and the string interactions are weak the D0-branes are described [7] by the minimal supersymmetric Yang-Mills (SYM) theory with sixteen supercharges. When the velocities or the string coupling are not small, this minimal supersymmetric theory is corrected by higher dimension operators. The suggestion of BFSS was that M theory in the infinite momentum frame in uncompactified space is obtained by considering the minimal SYM quantum mechanical system in the limit N, R_s, $P \to \infty$. For compactification on T^p the proposal of [1,2] is to consider Dp-branes, which are described by SYM in $p + 1$ dimensions, and again to truncate to the minimal theory. This proposal raised a few questions: (1) Why is this proposal correct? (2) Why is the theory with small R_s related to the theory with large R_s? (3) More specifically, the minimal supersymmetric theory is corrected by higher dimension operators, which are important when R_s and the velocities are not small. Why is the extrapolation from the minimal theory, which is valid at small R_s, correct for large R_s? (4) Furthermore, for $p \geq 4$ the minimal theory is not renormalizable and hence it is ill defined. Then, higher dimension operators must be included in the description. They reflect the fact that the theory must be embedded in a larger theory with more degrees of freedom. This theory for $p = 4, 5$ was found in [3–5]. The procedure to find these extension of the minimal theories did not appear systematic. What is then the rule to construct the theory in different backgrounds?

Susskind noted that the finite N matrix model enjoys some of the properties expected to hold only in the large N limit and suggested that it is also physically meaningful [8]. He suggested that the matrix model describes M theory compactified on a *lightlike* circle of radius R with momentum $P^+ = N/R$. Such a compactification on a lightlike circle with finite momentum is known as the discrete light cone and the quantization of this theory is known as the discrete light-cone quantization (DLCQ). With a lightlike circle the value of R can be changed by a boost. Therefore, the uncompactified theory cannot be obtained by simply taking R to infinity. Instead, it is obtained by taking $R, N \to \infty$, holding P^+ fixed.

In this paper we will relate these two approaches to the matrix theory. In the process of doing so, we will derive the matrix model and will answer the questions above. We will also present a uniform derivation of the matrix model for M theory on a compactified transverse space.

We start by reviewing some trivial facts about relativistic kinematics. A compactification on a lightlike circle corresponds to the identification

$$\begin{pmatrix} x \\ t \end{pmatrix} \sim \begin{pmatrix} x \\ t \end{pmatrix} + \begin{pmatrix} \frac{R}{\sqrt{2}} \\ -\frac{R}{\sqrt{2}} \end{pmatrix}, \tag{1}$$

where x is a spatial coordinate, e.g., x^{10}. We consider it as the limit of a compactification on a spacelike circle which is almost lightlike,

$$\begin{pmatrix} x \\ t \end{pmatrix} \sim \begin{pmatrix} x \\ t \end{pmatrix} + \begin{pmatrix} \frac{R^2}{2} + R_s^2 \\ -\frac{R}{\sqrt{2}} \end{pmatrix} \approx \begin{pmatrix} x \\ t \end{pmatrix} + \begin{pmatrix} \frac{R}{\sqrt{2}} + \frac{R_s^2}{\sqrt{2}R} \\ -\frac{R}{\sqrt{2}} \end{pmatrix}, \tag{2}$$

with $R_s \ll R$. The lightlike circle (1) is obtained from (2) as $R_s \to 0$. This compactification is related by a large boost with

$$\beta = \frac{R}{\sqrt{R^2 + 2R_s^2}} \approx 1 - \frac{R_s^2}{R^2} \tag{3}$$

to a spatial compactification on

$$\begin{pmatrix} x \\ t \end{pmatrix} \sim \begin{pmatrix} x \\ t \end{pmatrix} + \begin{pmatrix} R_s \\ 0 \end{pmatrix}. \tag{4}$$

A longitudinal boost of the lightlike circle (1) rescales the value of R. It also rescales the value of the light-cone energy P^-. Therefore P^- is proportional to R. For small R_s the value of P^- in the system with the almost lightlike circle (2) is also proportional to R (an exception to that occurs when $P^- = 0$ for the lightlike circle; then P^- can be nonzero for the almost lightlike circle). The boost (3) rescales P^- to be independent of R and of order R_s (if originally $P^- = 0$, the resulting P^- after the boost can be smaller than order R_s).

Following [9] (as referred to in [10]) we now consider M theory compactified on a lightlike circle (1) as the $R_s \to 0$ limit of the compactification on an almost lightlike circle (2) or as the limit of the boosted circle (4). This way the DLCQ of M theory discussed in [8] is related to the compactification on a small spatial circle as in [1]. For small R_s the theory compactified on (4) is the weakly coupled string theory with string coupling $g_s = (R_s M_P)^{3/2}$, and string scale $M_s^2 = R_s M_P^3$ (M_P is the Planck mass). For fixed energies and fixed M_P, the limit $R_s \to 0$ yields a complicated theory with vanishing string scale.

However, as mentioned above, starting with P^- of order one, the effect of the boost is to reduce P^- to be of order $R_s M_P^2$ (M_P^2 is inserted on dimensional grounds). This is exactly the range of energies in the discussion of [11], which was one of the motivations for the matrix model of BFSS [1]. In order to focus on the modes with such values of P^-, we rescale the parameters of the theory. We do that by replacing the original M theory, which is compactified on a lightlike circle of radius R, by another M theory, referred to as the $\tilde{M}$ theory with Planck scale M_P compactified on a spatial circle of radius R_s. The transverse geometry of the original M theory is replaced by that of the $\tilde{M}$ theory. For example, for a compactification on a transverse torus with radii R_i the other theory has radii $\tilde{R}_i$.

The relations between the parameters of these two theories are obtained by combining the limit $R_s \to 0$ with $\tilde{M}_P \to \infty$, holding $P^- \sim R_s \tilde{M}_P^2$ fixed. Therefore we identify

$$R_s \tilde{M}_P^2 = R M_P^2 \,, \tag{5}$$

which is finite in the limit. Since the boost does not affect the transverse directions, we identify

$$M_P R_i = \tilde{M}_P \tilde{R}_i \tag{6}$$

and keep it fixed. In this limit the energies are finite, and we find string theory with string coupling and string scale

$$\begin{aligned} \tilde{g}_s &= (R_s \tilde{M}_P)^{3/2} = R_s^{3/4} (R M_P^2)^{3/4}, \\ \tilde{M}_s^2 &= R_s \tilde{M}_P^3 = R_s^{-1/2} (R M_P^2)^{3/2}. \end{aligned} \tag{7}$$

For $R_s \to 0$ with finite M_P and R, we recover weakly coupled string theory with large string tension. This theory is very simple and is at the root of the simplification of the matrix model.

A sector with $P^+ = N/R$ in the original M theory is mapped to a sector of momentum $P = N/R_s$ in the new $\tilde{M}$ theory. In terms of this latter theory it includes N D0-branes. Therefore, the original M theory is mapped to the theory of D0-branes. These D0-branes move in a small transverse space of size $\tilde{R}_i \sim R_s^{1/2} \to 0$ (it is small even relative to the string length $\tilde{R}_i \tilde{M}_s \sim R_s^{1/4} \to 0$).

We conclude that the M theory with Planck scale M_P compactified on a lightlike circle of radius R and momentum $P^+ = N/R$ is the same as the $\tilde{M}$ theory with Planck scale $\tilde{M}_P$ compactified on a spatial circle (4) of radius R_s with N D0-branes in the limit

$$\begin{aligned} &R_s \to 0 \,, \\ &\tilde{M}_P \to \infty \,, \\ &R_s \tilde{M}_P^2 = R M_P^2 = \text{fixed}, \\ &\tilde{M}_P \tilde{R}_i = M_P R_i = \text{fixed}. \end{aligned} \tag{8}$$

Here R_i should be understood as generic parameters in the transverse metric—not only radii in a toroidal compactification. Clearly, the general discussion applies to curved space with any number of unbroken supercharges.

For a compactification on T^p we can use T duality to map the system of N D0-branes to N Dp-branes on a torus with larger radii

$$\Sigma_i = \frac{1}{\tilde{R}_i \tilde{M}_s^2} = \frac{1}{R_i R M_P^3} \,. \tag{9}$$

Note that Σ_i are finite when $R_s \to 0$. The string coupling after this T duality transformation is

$$\tilde{g}_s' = \tilde{g}_s \tilde{M}_s^p \prod \Sigma_i = \tilde{M}_s^p R^3 M_P^6 \prod \Sigma_i \,. \tag{10}$$

The low energy dynamics of these N Dp-branes are controlled by $(p+1)$-dimensional SYM [7] with gauge coupling

$$g_{\rm YM}^2 = \frac{\tilde{g}_s'}{\tilde{M}_s^{p-3}} = R^3 M_P^6 \prod \Sigma_i \,, \tag{11}$$

which also has a finite $R_s \to 0$ limit. Even though the low energy dynamics of these Dp-branes is finite in this limit, we should explore the behavior at higher energies. We will do that shortly.

In summary, we have mapped the original M theory problem with a lightlike circle of radius R and parameters M_P and R_i in the sector with $P^+ = N/R$ to a problem of N Dp-branes wrapping a torus in string theory. The radii of the torus, the string coupling, and string scale are

$$\begin{aligned} \Sigma_i &= \frac{1}{R_i R M_P^3} \,, \\ \tilde{g}_s' &= \tilde{M}_s^{p-3} R^3 M_P^6 \prod \Sigma_i \,, \\ \tilde{M}_s^2 &= R_s^{-1/2} (R M_P^2)^{3/2}, \end{aligned} \tag{12}$$

and $R_s \to 0$. Exactly this limit was analyzed recently in [12].

Let us analyze this limit for various values of p. For $p = 0$ the T duality which we performed is not necessary. The theory is that of D0-branes with vanishing string coupling and infinite string tension. Since the gauge coupling $g_{\rm YM}$ is finite, the theory is not trivial. The relevant degrees of freedom are strings stretched between the D0-branes. The infinite string scale decouples all the oscillators on the strings. Therefore the full theory is the minimal SYM theory. Note that closed strings or gravitons in the bulk of space-time decouple both because the string scale becomes large and because the string coupling vanishes. This is exactly the finite N version of the matrix model of BFSS [1].

For $p = 1, 2, 3$, we recover the SYM prescription of [1,2]. Again, the infinite string scale decouples the oscillators on the strings which are stretched between the N Dp-branes. Therefore the Lagrangian is that of the minimal SYM theory without higher order corrections. For $p = 3$ the string coupling $\tilde{g}_s'$ does not vanish, but there are still no higher dimension operators in the $(3 + 1)$-dimensional SYM, since they are all suppressed by inverse powers of the string scale $\tilde{M}_s$.

For $p = 4$ several new complications arise. First, the low energy SYM theory is not renormalizable and therefore cannot give a complete description of the theory. It breaks down at energies of order $1/g_{\rm YM}^2$, where new degrees of freedom must be added. Second, the string coupling also diverges in our limit. Therefore, in order to analyze the system we need to study the strong coupling limit of the $\tilde{M}$ theory, which is an eleven-dimensional theory. In this limit the D4-branes become 5-branes wrapping the eleventh dimension. Using (12), we find that this eleven-dimensional theory is compactified on a circle of finite radius

$$\Sigma_5 = \frac{\tilde{g}_s'}{\tilde{M}_s} = R^3 M_P^6 \prod \Sigma_i \,, \tag{13}$$

but its eleven-dimensional Planck scale diverges as

$$\frac{\tilde{M}_s}{(\tilde{g}_s')^{1/3}} \sim R_s^{-1/6} \to \infty \,. \tag{14}$$

Since the eleven-dimensional Planck scale is infinite, the modes in the bulk of space-time decouple, and the theory on the brane is a $(5 + 1)$-dimensional theory. This nontrivial theory, known as the $(2, 0)$ field theory, was first found in Refs. [13,14]. The new degrees of freedom, which we had to add at the energy of order $1/g_{\rm YM}^2$, can now be interpreted as associated with momentum modes around the circle (13). These are related to instantons in the SYM theory, which are D0-branes in the $\tilde{M}$ theory. We have thus derived the proposal of [3,4] to use this theory as a matrix theory for the M theory on T^4.

A similar analysis applies to $p = 5$. Here we study the strong coupling limit of D5-branes in IIB string theory. Using S duality of this theory, we map it to NS5-branes in weakly coupled IIB theory. We thus recover the proposal of [4,5] for the description of the M theory on T^5 in terms of a new theory obtained by studying NS5-branes in type-II theory. This theory, which can be called a noncritical string theory, is not a local quantum field theory. In addition to the five sides of the torus Σ_i (9), it is characterized by the string slope $\alpha' = g_{\rm YM}^2 = R^3 M_P^6 \prod \Sigma_i$. The SYM description breaks down at energies of order $1/g_{\rm YM}$, where new degrees of freedom are added. These degrees of freedom are the strings in the theory. In terms of the $\tilde{M}$ theory and its type-IIB string theory, these are D1-branes.

For $p = 6$ the situation is more complicated [15]. Here we are led to consider the strong coupling limit of D6-branes [16]. As for $p = 4$, this limit is described by an eleven-dimensional theory. However, here the eleven-dimensional Planck scale,

$$\frac{\tilde{M}_s}{(\tilde{g}_s')^{1/3}} = \frac{1}{R M_P^2 (\prod \Sigma_i)^{1/3}} \,, \tag{15}$$

remains finite, but the radius of the eleventh dimension diverges,

$$\frac{\tilde{g}_s'}{\tilde{M}_s} \sim R_s^{-1/2} \to \infty \,. \tag{16}$$

Since the radius diverges, the D6-branes, which are Kaluza-Klein monopoles associated with the eleventh dimension, expand and become an A_{N-1} singularity. The gauge coupling of the associated SYM theory is given by the eleven-dimensional Planck scale, which remains finite. Since the eleven-dimensional Planck scale is finite, there is no reason to assume that the gravitons in the bulk of the asymptotically locally Euclidean space with an A_{N-1} singularity decouple from it (see the discussion in Refs. [17–19]). From the $\tilde{M}$ theory point of view (before the T duality transformation), these graviton modes can be identified as Kaluza-Klein monopoles [20], which wrap the small T^6. Their energy is of order R_s^2. After the boost (3) their energy is of order R_s, and it vanishes as $R_s \to 0$. Therefore, the DLCQ theory has an infinite number of new massless modes.

We conclude that the M theory on T^6 and a lightlike circle with momentum $P^+ = N/R$ is the same as the M theory with Planck scale (15) compactified on T^6 with an A_{N-1} singularity in the noncompact spatial directions. The eleven-dimensional gravitons propagate in the entire space. There is also an SU(N) SYM in $6 + 1$ dimensions describing the interactions of gluons at the singularity. Compared with the situation for lower values of p, we seem to miss a decoupled $(6 + 1)$-dimensional U(1) multiplet. However, considering the limit which leads to this configuration carefully, we see that the U(1) multiplet exists. It is "smeared" over the four-dimensional noncompact space.

Unfortunately, this result is not satisfying. The matrix theory offered a simple description of M theory, which can lead to useful computations. Here we see that, for $p = 6$, it goes over to a situation which is apparently as complicated as the underlying *M* theory.

After completion of this work we received a paper [21] which partially overlaps ours. We also learned that J. Polchinski had independently reached some of these conclusions.

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[1] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, Phys. Rev. D **55**, 5112 (1997).
[2] W. Taylor IV, Phys. Lett. B **394**, 283 (1997).
[3] M. Rozali, Phys. Lett. B **400**, 260 (1997).
[4] M. Berkooz, M. Rozali, and N. Seiberg, Report No. hep-th/9704089.
[5] N. Seiberg, Report No. hep-th/9705221.
[6] J. Polchinski, Report No. hep-th/9611050; J. Polchinski, S. Chaudhuri, and C. Johnson, Report No. hep-th/9602052.
[7] E. Witten, Nucl. Phys. **B460**, 335 (1996).
[8] L. Susskind, Report No. hep-th/9704080.
[9] S. Hellerman and J. Polchinski (to be published).
[10] K. Becker, M. Becker, J. Polchinski, and A. Tseytlin, Phys. Rev. D **56**, 3174 (1997).
[11] M. Douglas, D. Kabat, P. Pouliot, and S. Shenker, Nucl. Phys. **B485**, 85 (1997).
[12] J. M. Maldacena, Report No. hep-th/9709099.
[13] E. Witten, Report No. hep-th/9507121.
[14] A. Strominger, Phys. Lett. B **383**, 44 (1996).
[15] O. Ganor, Report No. hep-th/9709139.
[16] We thank E. Witten for very helpful discussions on this limit.
[17] A. Losev, G. Moore, and S. Shatashvili, Report No. hep-th/9707250.
[18] I. Brunner and A. Karch, Report No. hep-th/9707259.
[19] A. Hanany and G. Lifschytz, Report No. hep-th/9708037.
[20] This interpretation is due to L. Susskind.
[21] A. Sen, Report No. hep-th/9709220.