Equipartition and Mass Segregation in a One-Dimensional Self-Gravitating System

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The simplest model for studying *N*-body gravitational interactions is the one-dimensional system consisting of parallel mass sheets [one-dimensional self-gravitating system (OGS)]. The model has been used often to test theories of gravitational evolution. Here we demonstrate that a two mass-component OGS with initial conditions selected far from equilibrium attains both mass segregation and equipartition of energy in a long, but finite, time. This may be the first clear evidence of the approach to thermal equilibrium in this type of system. The implications of these findings as well as future investigations are discussed. [S0031-9007(97)04431-1]

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The one-dimensional self-gravitating system (OGS) has been used as a simple model to study relaxation in *N*-body gravitational systems for several decades. In the OGS the gravitational field is uniform, and therefore simple algebraic equations which can be easily and rapidly solved on a computer govern the motion of the particles. Current computer technology allows for very long time dynamical simulations of reasonably large systems with little loss of numerical accuracy. Because the phase space of the OGS is compact, these systems do not suffer from some of the difficulties encountered in three dimensions (e.g., singularities, evaporation). However, a consequence of this is a somewhat tenuous connection with real galactic systems.

Computer simulations of the OGS show that they tend to progress through various quasiequilibrium states as they evolve from arbitrary initial conditions. These quasiequilibrium states often last for very long times, and are approximately stationary. Fluctuations caused by changes in the mean field potential within the initial nonstationary distribution rapidly decay and the system reaches a state of microscopic relaxation, distinguished from the longer macroscopic time scale for relaxation to thermal equilibrium. Early predictions and dynamical simulations put thermal equilibrium on a time scale proportional to $N^2 t_c$, where N is the number of particles and t_c is the typical time for a particle to traverse the system [1,2]. Later work tended to refute these earlier claims by showing that the system was still far from equilibrium after $2N^2 t_c$ [3,4]. Studies of the divergence of nearby trajectories in phase space using Lyapunov exponents predicted times much longer than $N^2 t_c$ for convergence [5]. More recent dynamical simulations have demonstrated that the relaxation time for arbitrary initial conditions, if it exists, is orders of magnitude greater than previously predicted [6–8]. Whether the OGS is capable of reaching a thermal equilibrium state from arbitrary initial conditions is still unclear.

Several recent papers have used equipartition of energy to investigate the approach to equilibrium of a single mass-species OGS [6–8]. They track the evolution of the system using a measure $\Delta(t)$ of the deviation of the average energy per particle from the theoretical equipartition value given by the virial theorem. After an initial short period of microscopic relaxation, $\Delta(t)$ tends to steadily decrease until a large peak appears. This peak, originally thought to indicate the onset of thermal equilibrium [6,7], was later seen as evidence that the system becomes hung up for long times in restricted (sticky) regions of the phase space [8]. Several additional peaks are typically seen following the initial large peak, indicating the continuing occurrence of transitions between quasiequilibrium states that closely mimic equilibrium and highly nonequilibrium states.

These recent works studying equilibrium of the OGS spurred our interest in studying a two mass-species version of the system. As a thermodynamic system approaches equilibrium, system members begin to share kinetic energy equally on the average. The equipartition theorem states that every coordinate or momentum which is represented by a simple quadratic expression in the Hamiltonian of a conservative system (e.g., $p^2/2m$) will, on the average, contribute $kT/2$ to the energy [9]. This average sharing of kinetic energy should be easily seen in a system containing two mass species. As the system evolves and the average kinetic energy of all particles equalize, the velocities of the heavy particles decrease, moving them toward the center of the system, and the light particle velocities increase, moving them outward, in a process known as mass segregation. The canonical ensemble predicts that the ratio of kinetic energies for the two mass species will approach one for a system in contact with an infinite reservoir. In an isolated system with a finite number of particles like the OGS (microcanonical ensemble), the equipartition theorem may only be exact in the limit of large N.

The attainment of mass segregation and equipartition of kinetic energy in a two mass-species OGS may provide the first clear, definitive, evidence of macroscopic relaxation toward thermal equilibrium. Because of limited computational ability, previous investigations of equipartition and mass segregation in the multiple mass-species OGS failed to reproduce the expected result [10–12]. We will see that the time scales employed in the earlier investigations were insufficient by orders of magnitude to observe true equipartition.

In this paper we describe dynamical simulations of a two mass-species, one-dimensional, self-gravitating system. The initial position and velocity of each particle is chosen by uniformly sampling a rectangular box in μ space, the (position, velocity) plane in which each particle is represented by a point. These conditions were selected because they provide an initial state far from equilibrium for which it is easy to characterize the kinetic energy ratio of the heavy and light particles. Since the locations in μ space of both heavy and light particles are sampled identically, there is no equipartition in the initial state and it is not typical of equilibrium. (For the case of a single mass system it is observed that the distribution in μ space will closely resemble a stationary solution to the Vlasov equation [13] after a short time.) In the canonical ensemble the average energy is $\frac{3}{4}$ in our units, and we choose an initial virial ratio $R_{\text{virial}} = 2(\text{kinetic energy}/\text{potential energy}) \approx 1$ to reduce the initial violent relaxation phase [14]. The average energy and virial ratio are set by properly choosing the size of the box. During the simulation, the time averaged kinetic energy, potential energy, and total energy of the two mass species are tracked as the system evolves. In addition, as a measure of mass segregation, the average magnitude of the distance from the center of mass of the system for each species is calculated. The simulations show that, following the initial period of violent relaxation [14], the average kinetic energy and distance for each mass species remain roughly constant for several $\times 10^6$ time units after which energy begins to transfer between mass species and equipartition and mass segregation set in.

Our two mass-species one-dimensional gravitating system is a collection of N planar sheets of constant mass density and infinite in, say, the x and y directions, that can move along the z direction under their mutual gravitational attraction. Each sheet can be considered a particle confined to move only along the z axis and the resulting gravitational field felt by each particle is uniform. Nothing other than gravitational forces are considered (no collisional terms), thus the particles do not collide but merely pass through (cross) one another. The system contains two species of particles with the heavy particles being 3 times the mass of the light ($m_H = 3m_L$). The system contains an equal number of each species ($N_L = N_H = N/2$). During a crossing, particles experience a discrete jump in their acceleration, while the sheet velocities remain continuous functions of time. Between crossings the particles simply undergo uniform acceleration produced by the inhomogeneity of the mass distribution. Because the system is isolated, momentum conservation allows us to fix the center of mass and set the total momentum to zero. The acceleration experienced by the jth particle from the left depends only on the difference between the total mass of particles to the right and the left and is given by

$$A_j = 2\pi G(M_R - M_L), \tag{1}$$

where M_R and M_L are the mass to the right and left of the jth particle, respectively, and G is the universal gravitational constant. The energy of the system is constant and is given by

$$E = \frac{1}{2}\sum_{j=1}^{N} m_j v_j^2 + 2\pi G \sum_{j<i} m_i m_j |x_i - x_j|, \tag{2}$$

where v_j and x_j are the velocity and position of the jth particle, respectively. In the remainder of the paper we employ dimensionless units scaled to the total system mass and energy [8,15]. The dimensionless unit of time τ corresponds to roughly one sixth of a characteristic crossing time, t_c.

We chose to simulate a system with 64 particles (mass sheets), 32 of each type. The initial state was obtained by uniformly sampling inside a rectangle in μ space to obtain a virial ratio of 1 and a system energy of approximately $\frac{3}{4}$. For the single mass system, it has been shown that this initial state will rapidly relax to a state that closely approximates a stationary "waterbag" solution to the Vlasov equation [6,7]. Here as well, there is evidence that the system rapidly enters a nearly stationary "Vlasov" state. A true Vlasov system will remain in the stationary waterbag state indefinitely. The discrete system tends to drift from one approximately stationary state to another

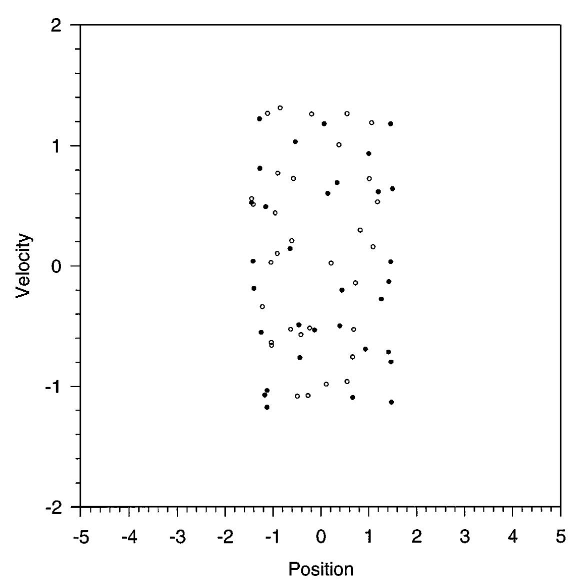

FIG. 1. The initial state of the system in μ space. The open circles are the light masses, and the filled circles are the heavy masses. All units are dimensionless.

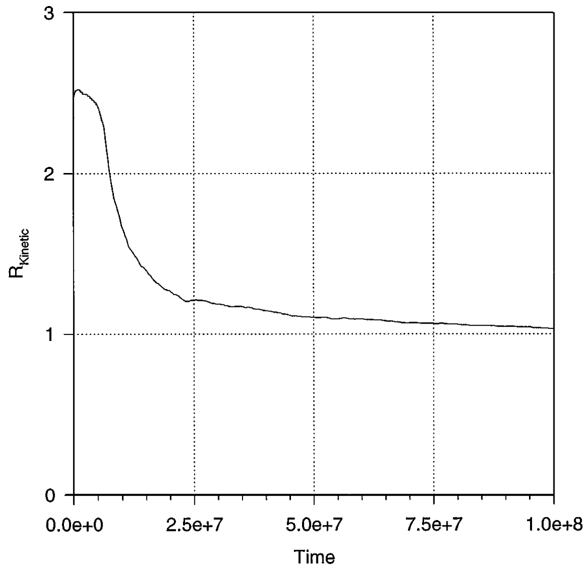

FIG. 2. The time average of R_{kinetic} showing an initial, roughly constant, value over a period of approximately $6 \times 10^6 \tau$, followed by a continual decrease toward $R_{\text{kinetic}} = 1$ over approximately $9 \times 10^7 \tau$. The final value of R_{kinetic} is 1.033. All units are dimensionless.

as the potential changes in time. The initial state of the system is plotted in Fig. 1.

Simulations were run for $1 \times 10^8 \tau$ using an exact code which solves for the time between crossings for each pair of particles. For these long computing runs, the energy is conserved to as good as 1 part in 10^{11}. Extreme care is taken not to lose numerical accuracy in the long time summations and averages required. For each component, the average distance from the center of mass of the system and the average kinetic energy are measured as the system evolves in time. Figure 2 shows the ratio of the kinetic energy for the two mass species, R_{kinetic}, plotted on a linear scale. As seen in the figure, R_{kinetic} remains reasonably constant for several $\times 10^6$ time units and then begins to change, moving downward toward $R_{\text{kinetic}} = 1$. Figure 3 shows the average distance from the center of mass for each species, D_H and D_L, plotted on a linear scale. After an initial, long, time period where D_H and D_L are approximately constant, they begin to change dramatically as the heavy masses start to condense toward the center of the system forming a core and the light masses move outward forming a halo in a process known as mass segregation.

In addition, starting at $t = 10^7 \tau$, we computed the average kinetic energy of each component as well as D_L for consecutive time segments of $6 \times 10^4 \tau$ for the remainder of the run. Using this data, we were able to construct mean values by averaging backwards in time from the end of the run (see Fig. 4).

This permitted the study of the system properties in the absence of the distortions produced by the initial, but long lasting, transient. It is clear from Fig. 4 that R_{kinetic} converges to unity. By comparing Fig. 5, the final positions of the system member in μ space at the end of the run, with Fig. 1, the segregation is apparent.

The simulation data presented in this short communication shows a clear change in the system dynamics after several $\times 10^6 \tau$. The equipartition of kinetic energy and

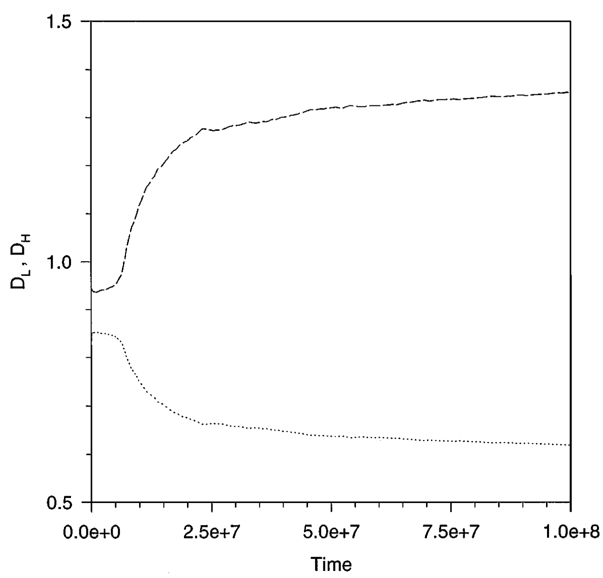

FIG. 3. The time average of D_L and D_H showing an initial, roughly constant period of approximately $6 \times 10^6 \tau$, followed by the onset of mass segregation of the light and heavy masses. The light masses are represented by the dashed line and the heavy masses are represented by the dotted line. All units are dimensionless.

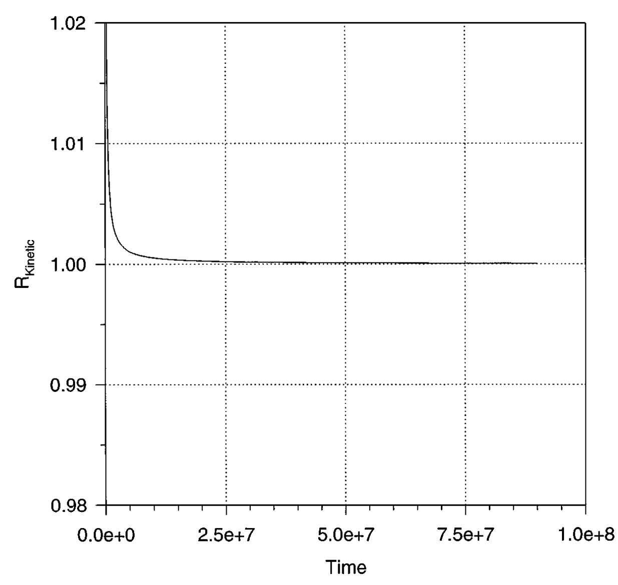

FIG. 4. The backward averaged R_{kinetic} over the last $9 \times 10^7 \tau$ showing a rapid decrease toward $R_{\text{kinetic}} = 1$. Transients caused by the initial non-Vlasov state are ignored. All units are dimensionless.

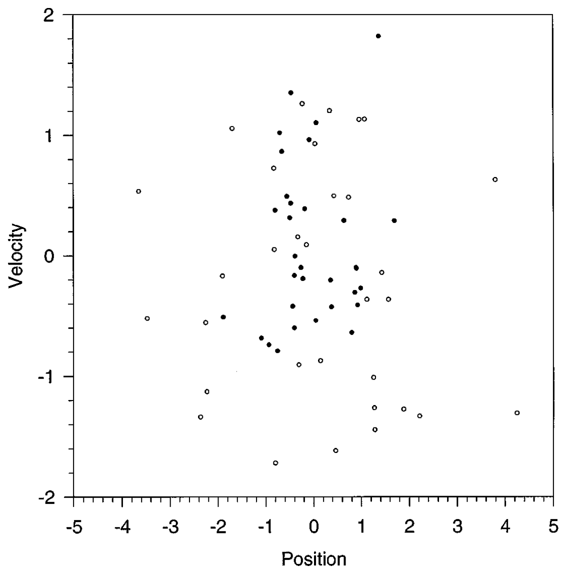

FIG. 5. The final state of the system in μ space showing mass segregation of the light and heavy masses. The open circles are the light masses and the filled circles are the heavy masses. All units are dimensionless.

mass segregation suggest an approach toward thermal equilibrium in a finite but long time. This is the first long time simulation of a multiple mass-species OGS and perhaps the first convincing evidence that this one-dimensional system can approach thermal equilibrium. The ratio of kinetic energies for the two species reaches a value of 1.033 after a time of $1 \times 10^8 \tau$.

The information gained from the sliding interval was also very useful. It allowed us to compute the mean of R_{kinetic} for the last $9.0 \times 10^7 \tau$ (90% of the run, see Fig. 4), as well as the spectrum of its fluctuations in each time segment. Thus we were able to remove the effect of the initial, but long lasting, transient. We obtained a value of 1.000 for the mean R_{kinetic} within the numerical accuracy of the experiment, showing that equilibrium was attained. In addition, we found that the variance was 0.0304, demonstrating that fluctuations were typically on the order of 20% in a given $6 \times 10^4 \tau$ time segment once equilibrium was attained.

With respect to the microcanonical ensemble, the spontaneous occurrence of the initial state of the system or other similar states is extremely unlikely. This can be seen quickly on the "back of an envelope" from the variance defined above. The occurrence of an $R_{\mathrm{kinetic}} \approx 3$ for a period of $6 \times 10^4 \tau$ is *over 17 standard deviations from the mean*. Recall, however, that the data show that the system dynamics "remembers" this state for several $\times 10^6 \tau$, i.e., for more than 20 of these time segments. This suggests that "memory" of the initial state controlled the behavior of all simulations of the system reported in the regular literature before 1994.

The backward averaging of the data provides unique information concerning the relaxation time of fluctuations. Figure 4 shows that fluctuations in R_{kinetic} relax on a time scale of $0.1 \times 10^7 \tau$, whereas Figs. 2 and 3 show that the initial macrostate persists for roughly 5–25 times longer. This should be contrasted with typical atomic and molecular systems in which there is a much stronger separation between microscopic and macroscopic relaxation processes.

In a follow-up to this Letter we will determine how the relaxation time for R_{kinetic} depends on the system population and the mass ratio, and the duration and patterns of large fluctuations in $R_{\mathrm{kinetic}}, D_H$, and D_L, for the two component OGS. If a pattern exists in these fluctuations, the time scales may match other features seen in the single component investigations that suggest the existence of small, residual, stable regions in the phase space which vanish with increasing N but are responsible for the slow approach of the timed averaged observables to their equilibrium values [8]. In addition, we have derived coupled differential equations that can be solved numerically for the canonical equilibrium probability densities of each component of the system in the Vlasov limit. From their solution, probability densities from dynamical simulations can be compared to those predicted by the Vlasov theory. Close agreement between theory and experiment would be a further indication of thermal equilibrium.

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