Instability and Depletion of an Excited Bose-Einstein Condensate in a Trap

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(Received 16 June 1997)

We present an analytical method to calculate the depletion of a Bose-Einstein condensate excited in a time dependent harmonic trap. We identify a regime where the motion of the condensate is unstable and show that this instability leads to an exponentially fast population of noncondensed modes. As these modes are concentrated on the surface of the condensate this is observable in the particle density. [S0031-9007(97)04428-1]

PACS numbers: 03.75.Fi, 05.30.Jp

Recently Bose-Einstein condensation has been demonstrated in dilute atomic gases [1]. Very low temperature condensates with up to 10^7 condensed particles can be prepared in magnetic traps. In this regime the interaction between the particles plays a significant role and has to be included in a theoretical treatment. The most widely used approach is a Hartree-Fock mean field approach which describes the state of the condensate by the Gross-Pitaevskii equation (GPE) [2]. The GPE neglects the possible contribution of noncondensed particles. An estimate of the fraction of noncondensed particles is possible with the Bogoliubov–de Gennes method [2]. For dilute gases at thermal equilibrium with temperatures well below the critical temperature the fraction of noncondensed particles is very small. In some nonequilibrium situations it may, however, increase exponentially with time. Studying how instability sets in when the condensate is driven hard as done in recent experiments [3] may help illuminate the onset of irreversibility in a quantum gas. We find that instability sets in for an atomic velocity on the order of the sound velocity. This is reminiscent of the destruction of superfluidity in liquid helium when the velocity of the liquid relative to the container is too large [2].

We consider a system with N particles. The existence of a macroscopically occupied state, the condensate, motivates splitting the atomic field operator $\hat{\Psi}$ as

$$\hat{\Psi}(\vec{r},t) = \Phi_{\rm ex}(\vec{r},t)\hat{a}_{\Phi_{\rm ex}} + \delta\hat{\Psi}(\vec{r},t)\,, \qquad (1)$$

where $\Phi_{\rm ex}$ is the *exact* condensate wave function and $\hat{a}_{\Phi_{\rm ex}}$ annihilates a particle in state $\Phi_{\rm ex}$. The fact that the remainder $\delta\hat{\Psi}(\vec{r},t)$ acting on the noncondensed particles has matrix elements $\sim\sqrt{N}$ times smaller than those of $\hat{a}_{\Phi_{\rm ex}}$ suggests an expansion in powers of the fraction of noncondensed particles. In [4] we derive such an expansion of the equations of motion of both the wave function $\Phi_{\rm ex}$ describing the dynamics of the condensed particles and the field operator $\hat{\Lambda}_{\rm ex} = \frac{1}{\sqrt{N}}\,\hat{a}^{\dagger}_{\Phi_{\rm ex}}\delta\hat{\Psi}$ describing the dynamics of the noncondensed particles [5]. The lowest order approximation Φ to $\Phi_{\rm ex}$ satisfies the time dependent GPE:

$$i\hbar\partial_t\Phi(\vec{r},t) = \mathcal{H}(t)\Phi(\vec{r},t)\,,$$

$$\mathcal{H}(t) = -\frac{\hbar^2\Delta}{2m} + \frac{m\omega^2(t)r^2}{2} + Ng|\Phi(\vec{r},t)|^2 - \mu\,, \qquad (2)$$

where μ is the chemical potential. We choose an isotropic harmonic trapping potential with time dependent frequency $\omega(t)$. The coupling constant g is related to the s-wave scattering length $a_s > 0$ by $g = 4\pi\hbar^2 a_s/m$. Initially the system is prepared in thermal equilibrium at temperature T much lower than the critical temperature. The state of the condensate is given by the lowest eigenstate $|\Phi_0\rangle$ of the time independent GPE, i.e., $\mathcal{H}(t=0)\Phi_0(\vec{r}) = 0$.

The time evolution of the noncondensed particles is generated by

$$i\hbar\partial_t\begin{pmatrix}\hat{\Lambda}(\vec{r},t)\\ \hat{\Lambda}^{\dagger}(\vec{r},t)\end{pmatrix} = \mathcal{L}(t)\begin{pmatrix}\hat{\Lambda}(\vec{r},t)\\ \hat{\Lambda}^{\dagger}(\vec{r},t)\end{pmatrix}, \qquad (3)$$

where $\hat{\Lambda}$ is the lowest order approximation to $\hat{\Lambda}_{\rm ex}$ and $\mathcal{L}$ a partial differential operator reminiscent of the Bogoliubov–de Gennes operator [4]:

$$\mathcal{L} = \begin{pmatrix}\mathcal{H} + NgQ|\Phi|^2Q & NgQ\Phi^2Q^*\\ -NgQ^*\Phi^{*2}Q & -\mathcal{H} - NgQ^*|\Phi|^2Q^*\end{pmatrix}. \qquad (4)$$

$Q(t) = \mathbb{1} - |\Phi(t)\rangle\langle\Phi(t)|$ projects orthogonally to Φ.

As shown in [4] the dynamics of the noncondensed particles is closely linked to the evolution generated by the GPE: A deviation $\delta\Phi$ from Φ will evolve according to the linearized GPE; remarkably the spinor $(\delta\Phi_\perp, \delta\Phi_\perp^*)$, where $\delta\Phi_\perp = Q(t)\delta\Phi$ is the deviation orthogonal to Φ, solves the same Eq. (4) as does $(\hat{\Lambda}, \hat{\Lambda}^{\dagger})$. We can therefore determine the dynamics of the noncondensed particles from a linear stability analysis of the GPE; in particular we will study the mean density of noncondensed particles $\langle\delta\hat{\Psi}^{\dagger}(\vec{r})\delta\hat{\Psi}(\vec{r})\rangle$ given to lowest order by

$$\delta\rho(\vec{r},t) \simeq \langle\hat{\Lambda}^{\dagger}(\vec{r},t)\hat{\Lambda}(\vec{r},t)\rangle\,. \qquad (5)$$

For an arbitrary $\omega(t)$ two scenarios can occur:

(1) The wave function $\Phi(t)$ is a stable solution of Eq. (2). The deviation $\delta\Phi_\perp(t)$ and therefore $\hat{\Lambda}$ grow at most polynomially with time in the case of marginal stability; $\delta\rho$ in Eq. (5) will follow the same type of law: No fast depletion of the condensate is expected.

(2) The wave function $\Phi(t)$ is unstable. This is in particular the case for chaotic motion of the condensate. $\delta\Phi_\perp(t)$ and $\hat{\Lambda}$ diverge exponentially with time and so will $\delta\rho$. For $\delta\rho \propto \exp(2\sigma t)$, where σ is a Liapunov

exponent, this leads to significant depletion after a time $\sim\log(N)/\sigma$.

Recent experiments [1] produce condensates well within the Thomas-Fermi regime $[\mu \gg \hbar\omega(0)]$, where one can neglect the kinetic energy in $\mathcal{H}$ compared to the particle interaction [6]. The subsequent evolution for a time dependent frequency $\omega(t)$ is approximated by a time dependent scaling and gauge transform for Φ [7,8] in Eq. (2), which absorbs the part of the interaction energy converted into kinetic energy. This generates a family of approximate solutions of Eq. (2):

$$\Phi_{\lambda(t)}(\vec{r}) = e^{-i\beta(t)} e^{imr^2\dot{\lambda}(t)/2\hbar\lambda(t)} \frac{\Phi_0(\{\vec{r}/\lambda(t)\})}{\sqrt{\lambda(t)^3}}. \quad (6)$$

We have $\hbar\dot{\beta} = \mu[1/\lambda^3 - 1]$ and for the scaling factor λ

$$\ddot{\lambda} = \frac{\omega^2(0)}{\lambda^4} - \omega^2(t)\lambda. \quad (7)$$

The condensate wave function is a member of this family; that is, $\Phi(t) = \Phi_{\lambda_0(t)}$ with $\lambda_0(0) = 1$ and $\dot{\lambda}_0(0) = 0$ as we impose $\Phi(t=0) = \Phi_0$.

To study how the density of noncondensed particles $\delta\rho$ may grow in time we have to solve Eq. (3). To do so we consider a neighboring solution of $\Phi(t) = \Phi_{\lambda_0(t)}$ within the family Eq. (6); that is, a solution $\Phi(t) = \Phi_{\lambda_0(t)+\delta\lambda(t)}$ with an arbitrarily small $\delta\lambda(t)$. As stated above the time evolution of both $\hat{\Lambda}$ and the orthogonal deviation $\delta\Phi_\perp(t) = Q(t)[\Phi_{\lambda_0(t)+\delta\lambda(t)} - \Phi_{\lambda_0(t)}]$ are generated by $\mathcal{L}(t)$. Linearizing Eq. (6) around $[\lambda_0(t), \dot{\lambda}_0(t)]$ we get

$$\begin{pmatrix} |\delta\Phi_\perp(t)\rangle \\ |\delta\Phi_\perp^*(t)\rangle \end{pmatrix} = \delta\lambda(t)\begin{pmatrix} |\eta(t)\rangle \\ |\eta^*(t)\rangle \end{pmatrix} + \delta\dot{\lambda}(t)\begin{pmatrix} |\zeta(t)\rangle \\ |\zeta^*(t)\rangle \end{pmatrix}, \quad (8)$$

a linear combination of two time dependent spinors with

$$\begin{aligned} |\eta(t)\rangle &= Q(t)\,(\partial_\lambda|\Phi_\lambda(t)\rangle)_{\lambda=\lambda_0}, \\ |\zeta(t)\rangle &= Q(t)\,(\partial_{\dot{\lambda}}|\Phi_\lambda(t)\rangle)_{\lambda=\lambda_0}. \end{aligned} \quad (9)$$

Equation (8) constitutes an explicit approximate solution of Eq. (3). The time evolution of the coefficients $\delta\lambda(t), \delta\dot{\lambda}(t)$ is obtained by linearizing Eq. (7) around λ_0.

Equation (8) can be generalized to the following class of approximate solutions to the time evolution generated by $\mathcal{L}(t)$ in Eq. (3):

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \frac{C(t)}{\chi}\begin{pmatrix} |\eta(t)\rangle \\ |\eta^*(t)\rangle \end{pmatrix} + \frac{\dot{C}(t)}{\chi}\begin{pmatrix} |\zeta(t)\rangle \\ |\zeta^*(t)\rangle \end{pmatrix}. \quad (10)$$

As does $\delta\lambda(t)$ the *complex* function $C(t)$ fulfills

$$\ddot{C}(t) = -\left[\frac{4\omega^2(0)}{\lambda_0^5(t)} + \omega^2(t)\right]C(t). \quad (11)$$

χ is an (at this stage) arbitrary normalization factor.

We investigate first the known case of a time independent trapping potential. Then $\lambda_0(t) \equiv 1$ and from Eq. (11) $\ddot{C} + \Omega_{\rm iso}^2 C = 0$ with $\Omega_{\rm iso} = \sqrt{5}\,\omega(0)$. The solutions $C(t) = \exp(\mp i\Omega_{\rm iso}t)$ give via Eq. (10) two eigenvectors of $\mathcal{L}(0)$ which describe the lowest isotropic excitation mode of the system [9]: the vector with energy $\hbar\Omega_{\rm iso}$, $[u_{\rm iso}(\vec{r},0), v_{\rm iso}(\vec{r},0)]$ and its peer $[v_{\rm iso}^*(\vec{r},0), u_{\rm iso}^*(\vec{r},0)]$ with opposite energy. The operator-valued components of the operator $\hat{\Lambda}$ on these vectors are such that

$$\begin{pmatrix} \hat{\Lambda}(\vec{r},0) \\ \hat{\Lambda}^\dagger(\vec{r},0) \end{pmatrix} = \hat{b}_{\rm iso}\begin{pmatrix} u_{\rm iso}(\vec{r},0) \\ v_{\rm iso}(\vec{r},0) \end{pmatrix} + \hat{b}_{\rm iso}^\dagger\begin{pmatrix} v_{\rm iso}^*(\vec{r},0) \\ u_{\rm iso}^*(\vec{r},0) \end{pmatrix} + \text{ all other modes}. \quad (12)$$

$\hat{b}_{\rm iso}, \hat{b}_{\rm iso}^\dagger$ annihilate/create an elementary excitation of the system with frequency $\Omega_{\rm iso}$ [2,4]. The bosonic commutation relation between $\hat{b}_{\rm iso}$ and $\hat{b}_{\rm iso}^\dagger$ imposes the normalization $\langle u_{\rm iso}|u_{\rm iso}\rangle - \langle v_{\rm iso}|v_{\rm iso}\rangle = 1$. This condition determines $\chi = 2m\Omega_{\rm iso}\hbar^{-1}\langle\Phi_0|r^2|\Phi_0\rangle$.

For a time dependent $\omega(t)$ the time evolution of Eq. (12) is obtained from Eqs. (10),(11) with the initial conditions $C(0) = 1, \dot{C}(0) = -i\Omega_{\rm iso}$ [10]. For a system initially in thermal equilibrium at temperature T, the contribution of the isotropic mode to $\delta\rho$ at time t is

$$\delta\rho_{\rm iso}(\vec{r},t) = (|u_{\rm iso}(\vec{r},t)|^2 + |v_{\rm iso}(\vec{r},t)|^2)\langle b_{\rm iso}^\dagger b_{\rm iso}\rangle + |v_{\rm iso}(\vec{r},t)|^2, \quad (13)$$

where $\langle b_{\rm iso}^\dagger b_{\rm iso}\rangle = (e^{\hbar\Omega_{\rm iso}/k_BT} - 1)^{-1}$. We note that a change in $\delta\rho_{\rm iso}$ is solely due to the time dependence of $u_{\rm iso}, v_{\rm iso}$.

The explicit calculation of $\delta\rho_{\rm iso}$ requires from Eq. (10) the evaluation of $\eta(r,t)$ and $\zeta(r,t)$, and therefore of $\Phi_0(r)$ and $\Phi_0'(r)$. In the Thomas-Fermi regime we get for $\Phi_0^2(\vec{r})$ an inverted parabola of spatial extension $r_0 = [2\mu/m\omega^2(0)]^{1/2}$. Finally, keeping only leading terms in the limit $\mu/\hbar\omega(0) \to \infty$, we get at $t = 0$:

$$\delta\rho_{\rm iso}(\vec{r},0) = \frac{15}{14}\frac{\mu}{\hbar\Omega_{\rm iso}}\Phi_0^2(r)\left(1 - \frac{7}{3}\frac{r^2}{r_0^2}\right)^2 \times\left[\langle b_{\rm iso}^\dagger b_{\rm iso}\rangle + \frac{1}{2}\right], \quad (14)$$

and at a later time $t > 0$:

$$\delta\rho_{\rm iso}(\vec{r},t) = \frac{|C(t)\dot{\lambda}_0(t) - \dot{C}(t)\lambda_0(t)|^2}{\Omega_{\rm iso}^2} \times \frac{\delta\rho_{\rm iso}(\frac{\vec{r}}{\lambda_0(t)},0)}{\lambda_0^3(t)}. \quad (15)$$

Using Eq. (15) we investigate the growth of $\delta\rho_{\rm iso}(\vec{r})$ for different time dependent frequencies $\omega(t)$:

In [3] the trapping frequency is modulated sinusoidally with a resonant frequency $\Omega_e \approx \Omega_{\rm iso}$ for a finite time

$$\omega(t) = \omega(0)[1 + \varepsilon\sin(\Omega_e t)] \qquad (0 < t < t_e) \quad (16)$$

and is restored to its initial value for $t > t_e$. Therefore the motion for the scaling factor $\lambda_0(t)$ for $t > t_e$ conserves the "breathing energy"

$$E = \frac{1}{2}\dot{\lambda}_0^2 + \frac{\omega^2(0)}{3\lambda_0^3} + \frac{1}{2}\omega^2(0)\lambda_0^2 - \frac{5}{6}\omega^2(0). \quad (17)$$

Consequently, the evolution of $\lambda_0(t)$ for $t > t_e$ is periodic, with a period $\tau(E)$. A neighboring trajectory $\lambda_0(t) + \delta\lambda(t)$ has a slightly different energy $E + \delta E$ and will oscillate with a slightly different period $\tau(E + \delta E) = \tau(E) + \tau'(E)\delta E$, so that its deviation from $\lambda_0(t)$ will increase linearly in time. In the limit $t - t_e \gg \tau(E)$ we obtain

$$C(t) \sim \frac{t - t_e}{\tau(E)} \tau'(E)\dot{\lambda}_0(t)(\ddot{\lambda}_0 C - \dot{\lambda}_0 \dot{C})(t_e). \quad (18)$$

The density in Eq. (15) diverges therefore only quadratically with time. For $\varepsilon \ll 1$ in Eq. (16), we find that $C(t)$ scales as ε^2, leading to a $\delta\rho_{\rm iso}$ scaling as $\varepsilon^4 t^2$ [11].

We have found that the isotropic mode does not lead to an exponential instability of the condensate for the above excitation scenario. To achieve instability in this mode, the most obvious scenario is a chaotic motion of the scaling factor in Eq. (7) for which $\delta\lambda$; that is, $C(t)$ and therefore $\delta\rho_{\rm iso}$ in Eq. (15) will diverge exponentially in time. In an isotropic trap chaotic motion can be obtained by *permanently* modulating the trap frequency with a strong amplitude, e.g., as $\omega(t) = \omega(0)[1 - \sin(\Omega_e t)/2]$, with a nonresonant frequency Ω_e. For $\Omega_e = 0.917\omega(0)$ a Poincaré section for the motion of $\lambda_0(t)$ exhibits regions of stochastic motion [12]. For an evolution time $\omega(0)t = 50$ we find from Eq. (15) an increase of $\delta\rho_{\rm iso}$ by a factor of approximately 10^3.

To get a complete understanding of the behavior of the system one has to take into account *all* excitation modes, as each of them can *a priori* become unstable. For an isotropic trap the complete set of modes is specified by the radial quantum number n and the angular quantum numbers l, m. We restrict the discussion to modes for which the Thomas-Fermi approximation of [9] applies. The eigenfrequencies are then given by $\Omega_{nl} = \hbar\omega[2n^2 + 2nl + 3n + l]^{1/2}$. Up to now we considered only the isotropic mode $n = 1, l = 0$ ($\Omega_{10} = \Omega_{\rm iso}$). To predict the dynamics of other modes we have to use a different approach. The idea is to directly solve the time evolution of spinors $[u(t), v(t)]$ generated by $\mathcal{L}(t)$ from Eq. (3) using the Thomas-Fermi approximation. First the unitary transform linking Φ_λ to Φ_0 in Eq. (6) is applied to u and v^*. This transforms Eq. (3). In the resulting equation the kinetic energy terms cannot be completely neglected, contrarily to the case of the GPE, but have to be included to first order. The detailed calculations will be presented elsewhere; we give here only the result.

Consider the eigenvector (u_{nlm}, v_{nlm}) of $\mathcal{L}(0)$ with the eigenenergy $\hbar\Omega_{nl}$; the evolution of the density of noncondensed particles in this mode can be obtained from the generalization of Eq. (15):

$$\delta\rho_{nlm}(\vec{r}, t) = \frac{|C_{nl}(t)\dot{\lambda}_0(t) - \dot{C}_{nl}(t)\lambda_0(t)|^2}{\Omega_{nl}^2} \times \frac{\delta\rho_{nlm}(\frac{\vec{r}}{\lambda_0(t)}, 0)}{\lambda_0^3(t)}, \quad (19)$$

where $C_{nl}(t)$ solves the linear equation

$$\ddot{C}_{nl}(t) = -\left[\omega^2(t) + \frac{\Omega_{nl}^2 - \omega^2(0)}{\lambda_0^5(t)}\right] C_{nl}(t) \quad (20)$$

with the initial conditions $C_{nl}(0) = 1, \dot{C}_{nl}(0) = -i\Omega_{nl}$. The initial density $\delta\rho_{nlm}(t = 0)$ is obtained from (u_{nlm}, v_{nlm}) in a way analogous to Eq. (13). Note that Eqs. (19) and (20) reduce to Eqs. (15) and (11) for the mode $n = 1, l = 0$. The modes $n = 0, l = 1$, with $\Omega_{01} = \omega(0)$ are the excitations of the center of mass motion; this is why Eq. (20) then reduces to the motion of a particle in the harmonic trap. Equation (20) involves only the eigenfrequency of the mode, not its spatial dependence; in consequence, modes with the same frequency exhibit the same stability behavior [13].

The degree of instability of C_{nl} in Eq. (20) is quantified by a Liapunov exponent σ_{nl}, such that C_{nl} diverges as $\exp(\sigma_{nl} t)$ when $t \to \infty$; the case of a polynomial divergence in time leads to $\sigma_{nl} = 0$. For a time dependent trapping frequency $\omega(t)$ equal to its initial value $\omega(0)$ for all $t > t_e$, we determine numerically σ_{nl} as a function of the breathing energy E defined in Eq. (17) for $t > t_e$. The result is shown in Fig. 1. For small E none of the modes are unstable. For large enough E several modes are unstable, the modes with the larger Liapunov exponents being $n = 0, l = 6$. At the onset of instability the local condensate velocity $v(\vec{r}) = r\dot{\lambda}_0/\lambda_0$ reaches a maximal value $\sim\omega(0)r_0$ which is close to the sound velocity $c = \omega(0)r_0/\sqrt{2}$ at $\vec{r} = 0, t = 0$.

In Fig. 2 we show $\delta\rho$ for different times with $\omega(t)$ varying as in Eq. (16); we neglect the contribution of modes with energy higher than μ, for which the

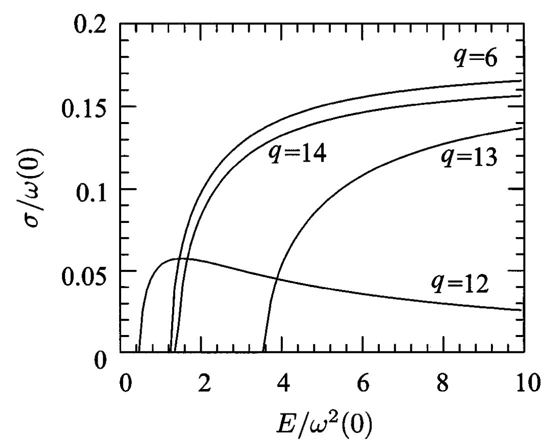

FIG. 1. For the evolution *after* the excitation phase $[t > t_e, \omega(t) = \omega(0)]$ Liapunov exponent as a function of the breathing energy E for the modes of frequency $\omega(0)q^{1/2}$, q integer ranging from 1 to 20 corresponding to modes (n, l) with $2n^2 + 2nl + 3n + l = q$. The values of q leading to a nonvanishing Liapunov exponent are indicated in the figure. The only modes corresponding to $q = 6$ are those with $n = 0, l = 6$.

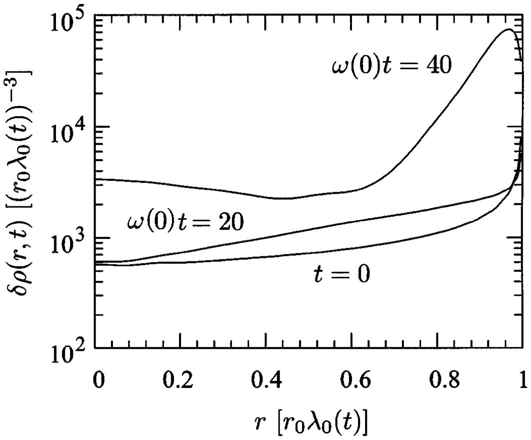

FIG. 2. For a sinusoidal modulation of $\omega(t)$ for five cycles [$\Omega_e = 5^{1/2}\omega(0)$, $t_e = 10\pi/\Omega_e$, $\varepsilon = 0.15$, leading to $E \simeq 3\omega^2(0)$], density of noncondensed particles at various times for the initial temperature $k_B T = 10\hbar\omega(0)$ and $\mu = 20\hbar\omega(0)$. The unit of length is $r_0\lambda_0(t)$, the spatial radius of the condensate at time t.

Thomas-Fermi approximation does not apply. In the long time limit, the dominant contribution comes from the modes $n = 0, l = 6$; as these are surface modes, they lead to a peak in the density of noncondensed particles close to the boundary of the condensate [14].

The above calculations can be extended to the anisotropic traps of [1]. The key property that we have used indeed is the existence of a scaling and gauge transform generating the evolution of the condensate wave function in the Thomas-Fermi limit. This property holds for harmonic traps with fixed eigenaxes [7,8].

In conclusion, we have identified experimentally accessible regimes where the motion of the condensate is unstable. This instability leads to an exponentially fast depletion of the condensate; as mainly surface modes of the condensate are then populated, this may result in observable changes of the spatial density of the particles.

We are grateful to M. Lewenstein, Ch. Miniatura, and M. Holzmann for useful discussions.

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[10] In Eq. (12) only the mode functions evolve in time: As $\mathcal{L}$ acts only on $\vec{r}$ dependent spinors the operators $\hat{b}_{\rm iso}$ are constants of motion [4].
[11] The scaling $\varepsilon^4 t^2$ does not apply for arbitrarily small εt: The first correction to the Thomas-Fermi approximation Eq. (6) contains terms of order ε correcting $\delta\Phi_\perp$ by a term scaling as εt; this term becomes negligible in the Thomas-Fermi limit when $\varepsilon\Omega_{\rm iso}t_e \gg 30(\Omega_{\rm iso} - \Omega_{\rm iso}^{\rm ex})/\Omega_{\rm iso}$ where $\Omega_{\rm iso}^{\rm ex}$ is the exact resonance frequency of the ground isotropic mode.
[12] As those regions are not immediately accessible from the initial conditions $\lambda_0 = 1, \dot{\lambda}_0 = 0$, the sinusoidal modulation of ω is preceeded by a preparation phase.
[13] For a finite value of $\mu/\hbar\omega(0)$ the mode frequencies differ from the values Ω_{nl} given in the text valid in the Thomas-Fermi limit $\mu/\hbar\omega(0) \to \infty$. We have therefore performed a numerical integration of Eqs. (2),(3); a good agreement with the analytical prediction from Eqs. (19),(20) is obtained only if Ω_{nl} is replaced by the exact value of the eigenfrequency in the calculation of C_{nl} in Eq. (20).
[14] As the directly measurable quantity is the total density ρ rather than $\delta\rho$, one should include the condensed fraction; as shown in [4] the first order deviation of $\Phi_{\rm ex}$ from Φ contributes to ρ to the same order as $\delta\rho$; this deviation is expandable on mode functions with $l = 0$ only, as $\Phi_{\rm ex}$ is isotropic, and therefore does not affect the peak of density observed for long times in Fig. 2.